**The parabolic Anderson model in a dynamic
random environment: space-time ergodicity
for the quenched Lyapunov exponent**

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Abstract

We continue our study of the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa \Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, and ξ plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$.

We assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log u(0,t)$. In earlier work [6], [3] we established a number of basic properties of $\kappa \mapsto \lambda_0(\kappa)$ under certain mild space-time mixing and noisiness assumptions on ξ. In particular, we showed that the limit exists ξ-a.s., is finite and continuous on $[0,\infty)$, is globally Lipschitz on $(0,\infty)$, is not Lipschitz at 0, and satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$.

In the present paper we show that $\lim_{\kappa \to \infty} \lambda_0(\kappa) = \mathbb{E}(\xi(0,0))$ under an additional space-time mixing condition on ξ we call Gärtner-hyper-mixing. This result, which completes our study of the quenched Lyapunov exponent for general ξ, shows that the parabolic Anderson model exhibits space-time ergodicity in the limit of large diffusivity. This fact is interesting because there are choices of ξ that are Gärtner-hyper-mixing for which the *annealed Lyapunov exponent* $\lambda_1(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(u(0,t))$ is infinite on $[0,\infty)$, a situation that is referred to as *strongly catalytic behavior*. Our proof is based on a *multiscale analysis* of ξ, in combination with discrete rearrangement inequalities for local times of simple random walk and spectral bounds for discrete Schrödinger operators.

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1   Introduction and main theorem

A fair amount is known about the behavior as a function of underlying parameters of the *annealed Lyapunov exponents* for the parabolic Anderson model in a dynamic random environment. For an overview we refer the reader to [5]. The main motivation behind the present paper is to understand the behavior of the *quenched Lyapunov exponent*, which is much harder to deal with. Our ultimate goal is to arrive at a full qualitative picture of the quenched Lyapunov exponent for *general* dynamic random environments subject to certain mild space-time mixing and noisiness assumptions.

Section 1.1 defines the parabolic Anderson model and recalls the main results from [6, 3]. Section 1.2 contains our main theorem, which states that the quenched Lyapunov exponent converges to the average value of the environment in the limit of large diffusivity. Section 1.3 contains definitions, whereas Section 1.4 discusses the main theorem, provides the necessary background, and gives a brief outline of the rest of the paper.

1.1   Parabolic Anderson model

The parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \xi(x,t) u(x,t), \qquad x \in \mathbb{Z}^d, \, t \geq 0. \tag{1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.2}$$

($\|\cdot\|$ is the l_1-norm), while

$$\xi = (\xi_t)_{t \geq 0} \text{ with } \xi_t = \{\xi(x,t) \colon x \in \mathbb{Z}^d\} \tag{1.3}$$

is an $\mathbb{R}$-valued random field playing the role of a *dynamic random environment* that drives the equation. As initial condition for (1.1) we take

$$u(x,0) = u_0(x), \, x \in \mathbb{Z}^d, \text{ with } u_0 \text{ non-negative, not identically zero, and bounded.} \tag{1.4}$$

One interpretation of (1.1) and (1.4) comes from *population dynamics*. Consider the special case where $\xi(x,t) = \gamma \xi_*(x,t) - \delta$ with $\delta, \gamma \in (0,\infty)$ and ξ_* an $\mathbb{N}_0$-valued random field. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

  – A-particles evolve autonomously according to a prescribed dynamics with $\xi_*(x,t)$ denoting the number of A-particles at site x at time t;

  – B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time;

  – B-particles die at rate δ;

  – the average number of B-particles at site x at time 0 is $u_0(x)$.

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Then
$$u(x,t) \quad = \quad \begin{array}{l} \text{the average number of } B\text{-particles at site } x \text{ at time } t \\ \text{conditioned on the evolution of the } A\text{-particles.} \end{array} \tag{1.5}$$

The ξ-field is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper we assume that

▶ ξ is *stationary* and *ergodic* under translations in space and time.

▶ ξ is *not constant* and $\mathbb{E}(|\xi(0,0)|) < \infty$. $\tag{1.6}$

The formal solution of (1.1) is given by the *Feynman-Kac formula*

$$u(x,t) = E_x \left(\exp\left\{ \int_0^t \xi(X^\kappa(s), t-s) \, ds \right\} u_0(X^\kappa(t)) \right), \tag{1.7}$$

where $X^\kappa = (X^\kappa(t))_{t \geq 0}$ is the continuous-time simple random walk jumping at rate $2d\kappa$ (i.e., the Markov process with generator $\kappa\Delta$), and P_x is the law of X^κ when $X^\kappa(0) = x$. In [3] we proved the following:

(0) Subject to the assumption that ξ-a.s. $s \mapsto \xi(x,s)$ is locally integrable for every x and that $\mathbb{E}(e^{q\xi(0,0)}) < \infty$ for all $q \geq 0$, (1.7) is finite for all x, t and is the solution of (1.1).

The *quenched Lyapunov exponent* associated with (1.1) is defined as

$$\lambda_0(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log u(0,t). \tag{1.8}$$

In [6] we showed that $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$ as soon as the limit in (1.8) exists. In [3] we proved the following:

(1) Subject to certain *space-time mixing assumptions* on ξ, the limit in (1.8) exists ξ-a.s. and in $L^1(\mathbb{P})$, is ξ-a.s. constant, is finite, and does not depend on u_0 satisfying (1.4).

(2) Subject to certain additional *noisiness assumptions* on ξ, $\kappa \mapsto \lambda_0(\kappa)$ is continuous on $[0,\infty)$, is globally Lipschitz on $(0,\infty)$, and is not Lipschitz at 0.

1.2   Main theorem and examples

Our main result is the following.

Theorem 1.1. *If $u_0 = \delta_0$ and ξ is Gärtner-hyper-mixing, then*

$$\lim_{\kappa \to \infty} \lambda_0(\kappa) = \mathbb{E}(\xi(0,0)). \tag{1.9}$$

The definition of Gärtner-hyper-mixing is given in Definitions 1.3–1.5 below. A weaker form of these definitions was introduced and exploited in [3]. Here are two examples of ξ-fields that are Gärtner-hyper-mixing.

Example 1.2. (See [3])
(e1) *Let $Y = (Y_t)_{t \geq 0}$ be a stationary and ergodic $\mathbb{R}$-valued Markov process satisfying*

$$\mathbb{E}\left[e^{q \sup_{t \in [0,1]} |Y_t|} \right] < \infty \qquad \forall q \geq 0. \tag{1.10}$$

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Let $(Y(x))_{x \in \mathbb{Z}^d}$ be a field of independent copies of Y. Then ξ given by $\xi(x, t) = Y_t(x)$ is Gärtner-hyper-mixing.

(e2) Let ξ be the zero-range process with rate function $g \colon \mathbb{N}_0 \to (0, \infty)$ given by $g(k) = k^\beta$, $\beta \in (0, 1]$, and transition probabilities given by simple random walk on $\mathbb{Z}^d$. If ξ starts from the product measure π_ρ, $\rho \in (0, \infty)$, with marginals

$$\forall x \in \mathbb{Z}^d \colon \quad \pi_\rho \{\eta \in \mathbb{N}_0^{\mathbb{Z}^d} \colon \eta(x) = k\} = \begin{cases} \gamma \, \frac{\rho^k}{\prod_{l=1}^k g(l)}, & \text{if } k > 0, \\ \gamma, & \text{if } k = 0, \end{cases} \tag{1.11}$$

where $\gamma \in (0, \infty)$ is a normalization constant, then ξ is Gärtner-hyper-mixing.

(The proof in [3] is without the supremum in (1.15) below, but easily carries over by inspection.) Example (e1) includes independent spin-flips, example (e2) includes independent random walks.

We expect that most interacting particle systems are Gärtner-hyper-mixing, including such classical systems as the stochastic Ising model, the contact process, the voter model and the exclusion process. Since these are bounded random fields, conditions (a2) and (a3) in Definition 1.5 below are redundant and only condition (a1) needs to be verified. Note that the constant δ in (1.15) below was allowed to be choosen arbitrarily large in [3]. However, in this work we assume that δ goes to zero in a certain way (see Definition (1.5)) so that (a1) indeed becomes an issue. We will not tackle the problem of solving this issue for the above mentioned fields in the present paper.

1.3 Definitions

Throughout the rest of this paper we assume without loss of generality that $\mathbb{E}(\xi(0, 0)) = 0$.

For $a_1, a_2, N \in \mathbb{N}$, denote by $\Delta_N(a_1, a_2)$ the set of $(\mathbb{Z}^d \times \mathbb{N})$-valued sequences $\{(x_i, k_i)\}_{i=1}^N$ that are increasing with respect to the lexicographic ordering of $\mathbb{Z}^d \times \mathbb{N}$ and are such that, for all $1 \leq i < j \leq N$,

$$x_j \equiv x_i \, (\mathrm{mod} \, a_1), \qquad k_j \equiv k_i \, (\mathrm{mod} \, a_2). \tag{1.12}$$

For $A \geq 1$, $\alpha > 0$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $k, b, c \in \mathbb{N}_0$, define the *space-time blocks* (see Fig. 1)

$$\tilde{B}_R^{A,\alpha}(x, k; b, c) = \left(\prod_{j=1}^d \left[(x(j) - 1 - b)\alpha A^R, (x(j) + 1 + b)\alpha A^R \right) \cap \mathbb{Z}^d \right) \times [(k-c)A^R, (k+1)A^R). \tag{1.13}$$

Abbreviate $B_R^{A,\alpha}(x, k) = \tilde{B}_R^{A,\alpha}(x, k; 0, 0)$ and $B_R^A(x, k) = B_R^{A,1}(x, k)$, and define the *space-blocks*

$$Q_R^{A,\alpha}(x) = x + [0, \alpha A^R)^d \cap \mathbb{Z}^d. \tag{1.14}$$

Definition 1.3. [Good and bad blocks]
For $A \geq 1$, $\alpha > 0$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $m > 0$, $k \in \mathbb{N}_0$, $\delta \in [0, \mathrm{ess\,sup}\,[\xi(0, 0)]]$ and $b, c \in \mathbb{N}_0$, the R-block $B_R^{A,\alpha}(x, k)$ is called (δ, b, c)-good for the potential ξ when, for all $s \in [(k - c)A^R, (k + 1)A^R - 1/m)$,

$$\frac{1}{|Q_R^{A,\alpha}(y)|} \sum_{z \in Q_R^{A,\alpha}(y)} \sup_{r \in [s, s+1/m)} \xi(z, r) \leq \delta \qquad \forall y \in \mathbb{Z}^d \colon Q_R^{A,\alpha}(y) \times \{s\} \subseteq \tilde{B}_R^{A,\alpha}(x, k; b, c), \tag{1.15}$$
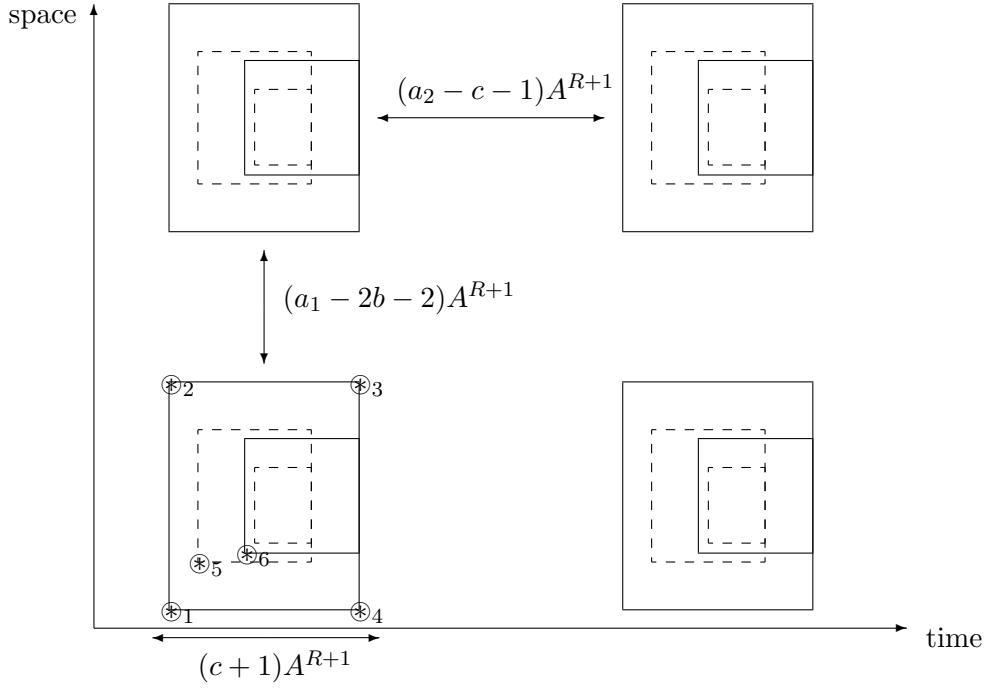
and is called (δ, b, c)-bad otherwise.

Figure 1: The dashed blocks are R-blocks, i.e., $B_R^A(x,k)$ (inner) and $\tilde{B}_R^A(x,k;b,c)$ (outer) for some choice of A,x,k,b,c. The solid blocks are $(R+1)$-blocks, i.e., $B_{R+1}^A(y,l)$ (inner) and $\tilde{B}_{R+1}^A(y,l;b,c)$ (outer) for some choice of A,y,l,b,c such that these $(R+1)$-blocks contain the corresponding R-blocks. All these blocks belong to the same equivalence class. The symbols $\{\circledast_i\}_{i=1,2,3,4,5,6}$ represents the space-time coordinates $\circledast_1 = ((y-1-b)A^{R+1}, (l-c)A^{R+1})$, $\circledast_2 = ((y+1+b)A^{R+1}, (l-c)A^{R+1})$, $\circledast_3 = ((y+1+b)A^{R+1}, (l+1)A^{R+1})$, $\circledast_4 = ((y-1-b)A^{R+1}, (l+1)A^{R+1})$, $\circledast_5 = ((x-1-b)A^R, (k-c)A^R)$, $\circledast_6 = ((y-1)A^{R+1}, lA^{R+1})$.

For $A \geq 1$, $\alpha > 0$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $m > 0$, $k \in \mathbb{N}_0$, $\delta \in [0, \operatorname{ess\,sup}[\xi(0,0)]]$ and $b,c \in \mathbb{N}_0$, let

$$
\begin{aligned}
&\mathcal{A}_R^{A,\alpha,\delta,m}(x,k;b,c) \\
&= \left\{ B_{R+1}^{A,\alpha}(x,k) \text{ is } (\delta,b,c)\text{-good, but contains an } R\text{-block that is } (\delta,b,c)\text{-bad} \right\}.
\end{aligned}
\tag{1.16}
$$

Definition 1.4. [Gärtner-mixing]
The ξ-field is called (A,α,δ,m,b,c)-Gärtner-mixing when there are $a_1, a_2 \in \mathbb{N}$ such that

$$
\sup_{(x_i,k_i)_{i=1}^N \in \Delta_N(a_1,a_2)} \mathbb{P}\left(\bigcap_{i=1}^N \mathcal{A}_R^{A,\alpha,\delta,m}(x_i,k_i;b,c) \right) \leq \left(A^{-4d(2d+1)(d+1)R} \right)^N \qquad \forall R \in \mathbb{N}, N \in \mathbb{N}.
\tag{1.17}
$$

Definition 1.5. [Gärtner-hyper-mixing]
The ξ-field is called Gärtner-hyper-mixing when the following conditions are satisfied:
(a1) There are $b,c \in \mathbb{N}_0$ and $K \geq 0$ such that for every $\delta > 0$ there are $A_0 > 1$ and $m_0 > 0$ such that $\xi \mathbb{1}\{\xi \geq K\}$ and ξ are (A,α,δ,m,b,c)-Gärtner-mixing for all $A \geq A_0$, $m \geq m_0$ and all $\alpha \geq 1$, with a_1, a_2 in Definition 1.4 not depending on A, m and α.
(a2) $\mathbb{E}[e^{q \sup_{s \in [0,1]} \xi(0,s)}] < \infty$ for all $q \geq 0$.
(a3) There are $R_0 \in \mathbb{N}$ and $C_1 \in [0, \operatorname{ess\,sup}[\xi(0,0)]]$ such that

$$
\mathbb{P}\left(\sup_{s \in [0,1]} \frac{1}{|B_R|} \sum_{y \in B_R} \xi(y,s) \geq C \right) \leq |B_R|^{-\alpha} \quad \forall R \geq R_0, C \geq C_1,
\tag{1.18}
$$

for some $\alpha > [2d(2d+1)+1](d+2)/d$, where $B_R = [-R,R]^d \cap \mathbb{Z}^d$.

1.4 Discussion

1. What is interesting about Theorem 1.1 is that it reveals a sharp contrast with what is known for the *annealed Lyapunov exponent*

$$\lambda_1(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(u(0, t)). \qquad (1.19)$$

Indeed, there are choices of ξ for which $\kappa \mapsto \lambda_1(\kappa)$ is everywhere infinite on $[0, \infty)$, a property referred to as *strongly catalytic* behavior. For instance, as shown in [4], if ξ is γ times a field of independent simple random walks starting in a Poisson equilibrium with arbitrary density, then this uniform divergence occurs in $d = 1, 2$ for $\gamma \in (0, \infty)$ and in $d \geq 3$ for $\gamma \in [1/G_d, \infty)$, with G_d the Green function of simple random walk at the origin. By Example 1.2(e2) (with $\beta = 1$), this choice of ξ is Gärtner-hyper-mixing.

2. The annealed Lyapunov exponents

$$\lambda_p(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}([u(0, t)]^p), \qquad p \in \mathbb{N}, \qquad (1.20)$$

were studied in detail in a series of papers where ξ was chosen to evolve according to four specific interacting particle systems in equilibrium: independent Brownian motions, independent simple random walks, the simple symmetric exclusion process, and the voter model (for an overview, see [5]). Their behavior turns out to be very different from that of $\lambda_0(\kappa)$. In [3] it was conjectured that

$$\lim_{\kappa \to \infty} [\lambda_p(\kappa) - \lambda_0(\kappa)] = 0 \qquad \forall \, p \in \mathbb{N} \qquad (1.21)$$

because ξ is ergodic in space and time. For the case where $\lambda_p(\kappa) \equiv \infty$ this statement is to be read as saying that $\lim_{\kappa \to \infty} \lambda_0(\kappa) = \infty$. Theorem 1.1 shows that this conjecture is false and that, for ξ Gärtner-hyper-mixing and satisfying conditions (0) and (2) in Section 1.1, the qualitative behavior of $\kappa \mapsto \lambda_0(\kappa)$ is as in Fig. 2.
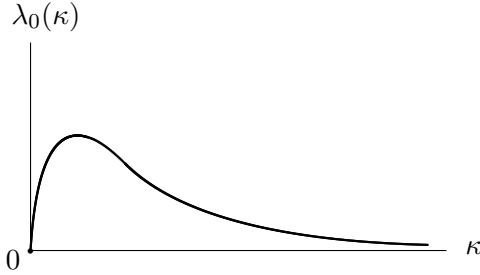

Figure 2: Qualitative behavior of $\kappa \mapsto \lambda_0(\kappa)$.

3. Our proof of Theorem 1.1 is based on a *multiscale analysis* of ξ, in the spirit of [10] and consists of two major steps:

(I) We look at the bad R-blocks for all $R \in \mathbb{N}$. First we show that bad R-blocks are rare for large R. Next, using a *discrete rearrangement inequality* for local times of simple random walk, we show that the contribution to the expectation in (1.7) coming from bad R-blocks increases when we move them towards the origin. Therefore this contribution can be bounded from above by an expectation that pretends the bad R-blocks to be

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rearranged in a space-time cylinder around the origin. Since bad R-blocks are rare, this cylinder is narrow. Afterwards, because simple random walk is unlikely to spend a lot of time in narrow space-time cylinder, we are able to control the contribution coming from bad R-blocks to the expectation (1.7) uniformly in t and κ.

(II) We look at the good R-blocks for all $R \in \mathbb{N}$. We control their contribution by using an *eigenvalue expansion* of (1.7). An analysis of the largest eigenvalue in this expansion concludes the argument.

The remainder of this paper is organized as follows. In Section 2 we formulate three key propositions and use these to prove Theorem 1.1. The three propositions are proved in Sections 3–7, respectively. In Appendix A we prove two technical lemmas that are needed in Section 4, while in Appendix B we prove a spectral bound that is needed in Section 7.

2 Three key propositions and proof of Theorem 1.1

To state our three key propositions we need some definitions. Fix $k_* \in \mathbb{N}$, and $t > 0$. We say that $\Phi \colon [0, t] \to \mathbb{Z}^d$ is a path when

$$\|\Phi(s) - \Phi(s-)\| \leq 1 \qquad \forall \, s \in [0, t]. \tag{2.1}$$

Define the set of paths

$$\Pi(k_*, t, A) = \big\{ \Phi \colon [0, t] \to \mathbb{Z}^d \colon \Phi \text{ crosses } k_* \text{ 1-blocks} \big\}. \tag{2.2}$$

For $B_R \subseteq \mathbb{Z}^d \times [0, t]$, let $l_t(B_R)$ denote the local time of X^κ in B_R up to time t, and $l_t(\pi_1(B_R))$ the local time of $\pi_1(X^\kappa)$ (the first coordinate of X^κ) in $\pi_1(B_R)$ up to time t. Furthermore, let $l_t(\mathrm{BAD}_R^\delta(\xi_K))$ and $l_t(\mathrm{BAD}^\delta(\xi))$ denote the local time of X^κ in (δ, b, c)-bad R-blocks up to time t for the potential $\xi \mathbb{1}\{\xi \geq K\}$ and in (δ, b, c)-bad 1-blocks up to time t for the potential ξ, respectively. Here and in the rest of the paper a bad R-block is (δ, b, c)-bad for a choice of K, δ, b, c and some $A \geq A_0$, $m \geq m_0$, according to Definition 1.5.

In what follows, when we write sums like $\sum_{0 \leq k < t/A^R}$ or $\sum_{R=1}^{\varepsilon \log t}$ we will pretend that t/A^R and $\varepsilon \log t$ are integer in order not to burden the notation with round off brackets. From the context it will always be clear where to place the brackets.

Proposition 2.1. *There is a $C_2 > 0$ such that for every $\varepsilon > 0$ and $\delta > 0$ there is an $A = A(\varepsilon) > 3$, satisfying $\lim_{\varepsilon \downarrow 0} A(\varepsilon, \delta) = \infty$, such that ξ-a.s. for all $\kappa > 0$ and all $t > 0$ large enough,*

$$E_0 \left(\exp\left\{ \int_0^t \xi(X^\kappa(s), t - s) \, ds \right\} \right) \leq e^{-t} + \left[E_0 \left(\exp\left\{ \int_0^t \overline{\xi}(X^\kappa(s), t - s) \, ds \right\} \right)^{1/2} \right.$$

$$\times E_0 \left(\exp\left\{ 2\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + 2 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\big(\mathrm{BAD}_R^\delta(\xi_K)\big) \right\} \right. \tag{2.3}$$

$$\left. \left. \times \, \mathbb{1}\big\{ \exists \, k_* \leq C_2 \kappa t \colon X^\kappa \in \Pi(k_*, t, A) \big\} \right)^{1/2} \right],$$

where

$$\overline{\xi}(x, s) = 2\xi(x, s) \mathbb{1}\big\{ \xi(x, s) < \delta A^d, (x, s) \text{ is in a good 1-block of } \xi \big\}. \tag{2.4}$$

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Proposition 2.2. *There is a $C_2 > 0$ such that for every $\varepsilon, \tilde{\varepsilon} > 0$ and $\delta > 0$ there is an $A = A(\varepsilon, \tilde{\varepsilon}, \delta) > 3$, satisfying $\lim_{\tilde{\varepsilon} \downarrow 0} A(\varepsilon, \tilde{\varepsilon}, \delta) = \infty$, such that ξ-a.s. for all $\kappa > 0$ and all $t > 0$ large enough,*

$$
E_0 \Bigg(\exp \Bigg\{ 2\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + 2 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t \big(\mathrm{BAD}_R^\delta(\xi_K) \big) \Bigg\}
$$
$$
\times \mathbb{1}\big\{ \exists\, k_* \leq C_2 \kappa t \colon\ X^\kappa \in \Pi(k_*, t, A) \big\} \Bigg) \leq e^{\tilde{\varepsilon} t}. \tag{2.5}
$$

Proposition 2.3. *There is a constant $C_3 > 0$ such that, for every $A > 1$ and $\delta > 0$,*

$$
\limsup_{\kappa \to \infty} \limsup_{n \to \infty} \frac{1}{An} \log E_0 \Bigg(\exp \Bigg\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \Bigg\} \Bigg) \leq \frac{C_3}{A} + 4\delta \qquad \xi - a.s. \tag{2.6}
$$

Proposition 2.1 estimates the Feynman-Kac formula in (1.7) in terms of bad blocks and good blocks, Proposition 2.2 controls the contribution of bad block, while Proposition 2.3 controls the contribution of good blocks.

We are now ready to prove Theorem 1.1.

Proof. Note that by Theorem 1.2(i) in [6], for all $\kappa \geq 0$ we have the lower bound $\lambda_0(\kappa) \geq 0$. Thus, it suffices to show the inequality in the reverse direction. To that end, fix $C_2, C_3 > 0$ according to Propositions 2.1–2.3, and fix $\varepsilon, \tilde{\varepsilon}, \delta > 0$. According to Proposition 2.2, there is an $A = A(\varepsilon, \tilde{\varepsilon}, \delta)$ such that, ξ-a.s. for all $\kappa > 0$ and all t of the form $t = An$ with $n \in \mathbb{N}$ large enough, the term in the left-hand side of (2.5) is bounded from above by $e^{\tilde{\varepsilon} An}$. According to Proposition 2.3, we have

$$
E_0 \Bigg(\exp \Bigg\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \Bigg\} \Bigg) \leq e^{4\delta An + C_3 n + \chi(\kappa, n)} \tag{2.7}
$$

with $\limsup_{\kappa \to \infty} \limsup_{n \to \infty} \chi(\kappa, n)/n = 0$. Proposition 2.1 therefore yields that, for all $\varepsilon, \tilde{\varepsilon}, \delta > 0$,

$$
\limsup_{\kappa \to \infty} \lambda_0(\kappa) \leq \frac{C_3}{2A} + 2\delta + \frac{\tilde{\varepsilon}}{2}. \tag{2.8}
$$

Since $\lim_{\tilde{\varepsilon} \downarrow 0} A(\varepsilon, \tilde{\varepsilon}, \delta) = \infty$ by Proposition 2.2, we get that for all $\delta > 0$,

$$
\limsup_{\kappa \to \infty} \lambda_0(\kappa) \leq 2\delta. \tag{2.9}
$$

Let $\delta \downarrow 0$ to get the claim. $\blacksquare$

3 Proof of Proposition 2.1

The proof is given in Section 3.1 subject to Lemmas 3.1–3.2 below. The proof of these lemmas is given in Section 3.2.

3.1 Proof of Proposition 2.1 subject to two lemmas

For $A \geq 1$, $R \in \mathbb{N}$ and $\Phi \in \Pi(k_*, t, A)$, define

$$
\begin{aligned}
\Xi_R^A(\Phi) &= \text{number of bad } R\text{-blocks crossed by } \Phi, \\
\Xi_R^{A,k_*} &= \sup_{\Phi \in \Pi(k_*,t,A)} \Xi_R^A(\Phi).
\end{aligned}
\tag{3.1}
$$

Lemma 3.1. *For every $\varepsilon > 0$ there is an $A = A(\varepsilon) > 3$ satisfying $\lim_{\varepsilon \downarrow 0} A(\varepsilon) = \infty$ such that*

$$
\mathbb{P}\Big(\Xi_R^{A,k_*} > 0 \text{ for some } R \geq \varepsilon \log t \text{ and some } k_* \in \mathbb{N}\Big)
\tag{3.2}
$$

is summable over $t \in \mathbb{N}$. A possible choice is $A = e^{1/a\varepsilon[2d(2d+1)+1]}$ for some $a > 1$.

Lemma 3.2. *There is a $C_2 > 0$ such that ξ-a.s. for all $A > 1$, all $t > 0$ and all $\kappa > 0$ large enough,*

$$
E_0\left(\exp\left\{ \int_0^t \xi(X^\kappa(s), t-s)\, ds \right\} \mathbb{1}\big\{ \exists\, k_* > C_2 \kappa t\colon\, X^\kappa \in \Pi(k_*, t, A) \big\} \right) \leq e^{-t}.
\tag{3.3}
$$

We are now ready to prove Proposition 2.1.

Proof. Fix C_2 in accordance with Lemma 3.2 and $\varepsilon > 0$. Let $\delta > 0$ and fix $A > 1$ according to Lemma 3.1 such that $\delta A^d \geq K$, see Definition 1.5. Note that

$$
\begin{aligned}
& E_0\left(\exp\left\{ \int_0^t \xi(X^\kappa(s), t-s)\, ds \right\} \mathbb{1}\big\{ \exists\, k_* \leq C_2 \kappa t\colon\, X^\kappa \in \Pi(k_*, t, A) \big\} \right) \\
&= E_0\Bigg(\exp\Bigg\{ \sum_{i=1}^{N(X^\kappa,t)} \int_{s_{i-1}}^{s_i} \xi(x_{i-1}, t-u)\, du + \int_{s_{N(X^\kappa,t)}}^t \xi(x_{N(X^\kappa,t)}, t-u)\, du \Bigg\} \\
&\qquad\qquad \times \mathbb{1}\big\{ \exists\, k_* \leq C_2 \kappa t\colon\, X^\kappa \in \Pi(k_*, t, A) \big\} \Bigg),
\end{aligned}
\tag{3.4}
$$

where $N(X^\kappa, t)$ is the number of jumps by X^κ up to time t, $0 = x_0, x_1, \ldots, x_{N(X^\kappa,t)}$ are the nearest-neighbor sites visited, and $0 = s_0 < s_1 < \cdots < s_{N(X^\kappa,t)} \leq t$ are the jump times. To analyze (3.4), define

$$
\begin{aligned}
\Lambda_t(\mathrm{BAD}_R^\delta) = \bigcup_{i=1}^{N(X^\kappa,t)} &\left\{ u \in [s_{i-1}, s_i)\colon\, \delta A^{Rd} < \xi(x_{i-1}, t-u) \leq \delta A^{(R+1)d} \right\} \\
&\bigcup \left\{ u \in [s_{N(X^\kappa,t)}, t)\colon\, \delta A^{Rd} < \xi(x_{N(X^\kappa,t)}, t-u) \leq \delta A^{(R+1)d} \right\}.
\end{aligned}
\tag{3.5}
$$

Then the contribution to the exponential in (3.4) may be bounded from above by

$$
\int_0^t \xi(X^\kappa(s), t-s) \mathbb{1}\{\xi(X^\kappa(s), t-s) < \delta A^d\}\, ds + \sum_{R \in \mathbb{N}} \delta A^{(R+1)d} \big| \Lambda_t(\mathrm{BAD}_R^\delta) \big|,
\tag{3.6}
$$

By Definition 1.3 and the fact that $\delta A^d \geq K$ (see the line preceding (3.4)), if $\delta A^{Rd} < \xi(x_{i-1}, t - u) \leq \delta A^{(R+1)d}$, then $(x_{i-1}, t-u)$ belongs to a bad R-block for the potential $\xi \mathbb{1}\{\xi \geq K\}$. Hence

$$
\big| \Lambda_t(\mathrm{BAD}_R^\delta) \big| \leq l_t(\mathrm{BAD}_R^\delta(\xi_K)).
\tag{3.7}
$$

To continue we write the indicator in (3.5)

$$
\begin{aligned}
&\mathbb{1}\{\xi(X^\kappa(s), t-s) < \delta A^d, (X^\kappa(s), t-s) \text{ is in a good 1-block of } \xi\} \\
&+ \mathbb{1}\{\xi(X^\kappa(s), t-s) < \delta A^d, (X^\kappa(s), t-s) \text{ is in a bad 1-block of } \xi\}
\end{aligned}
\tag{3.8}
$$

By Lemma 3.1 and our choice of A at the beginning of the proof, ξ-a.s. for t large enough there are no bad R-blocks with $R > \varepsilon \log t$. Thus, the expectation in the right-hand side of (3.4) may be estimated from above by

$$
\begin{aligned}
E_0\Bigg(&\exp\Bigg\{ \int_0^t \xi(X^\kappa(s), t-s) \\
&\times \mathbb{1}\{\xi(X^\kappa(s), t-s) < \delta A^d, (X^\kappa(s), t-s) \text{ is in a good 1-block of } \xi\}\, ds \Bigg\} \\
&\times \exp\Bigg\{ \delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R^\delta(\xi_K)) \Bigg\} \\
&\times \mathbb{1}\{\exists\, k_* \le C_2 \kappa t\colon X^\kappa \in \Pi(k_*, t, A)\} \Bigg).
\end{aligned}
\tag{3.9}
$$

Recall (2.4). An application of the Cauchy-Schwarz inequality yields the following upper bound for (3.9):

$$
\begin{aligned}
E_0\Bigg(&\exp\Bigg\{ \int_0^t \overline{\xi}(X^\kappa(s), t-s)\, ds \Bigg\} \Bigg)^{1/2} \\
&\times E_0\Bigg(\exp\Bigg\{ 2\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + 2\sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R^\delta(\xi_K)) \Bigg\} \\
&\qquad\qquad \times \mathbb{1}\{\exists\, k_* \le C_2 \kappa t\colon X^\kappa \in \Pi(k_*, t, A)\} \Bigg)^{1/2}.
\end{aligned}
\tag{3.10}
$$

The claim in (2.3) therefore follows by combining (3.4), (3.6–3.7) and (3.9)–(3.10) with Lemma 3.2. ∎

3.2   Proof of Lemmas 3.1–3.2

Proof. For the proof of Lemma 3.1, see [3, Lemma 3.3]. To prove Lemma 3.2, use Cauchy-Schwarz to estimate the expectation in (3.3) from above by

$$
\left[E_0\Bigg(\exp\Bigg\{ 2\int_0^t \xi(X^\kappa(s), t-s)\, ds \Bigg\} \Bigg) \right]^{1/2} \left[P_0\Big(\exists\, k_* > C_2 \kappa t\colon X^\kappa \in \Pi(k_*, t, A) \Big) \right]^{1/2}.
\tag{3.11}
$$

To bound the first term in (3.11), note that by [3, Eq.(3.54)] there is a $C > 0$ such that ξ-a.s. for all $t, \kappa > 0$,

$$
E_0\Big(e^{2\int_0^t \xi(X^\kappa(s), t-s)\, ds} \Big) \le e^{tC(\kappa+1)}.
\tag{3.12}
$$

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To bound the second term in (3.11) we use a similar strategy as for the proof of Lemma 4.4. Given $l_1, \ldots, l_{t/A} \in \mathbb{N}$, we say that X^κ has label $(l_1, \ldots, l_{t/A})$ when X^κ crosses l_i 1-blocks in the time interval $[(i-1)A, iA)$, $i \in \{1, \ldots, t/A\}$. Fix $C_2 > 0$ and write

$$
P_0\left(X^\kappa \in \Pi(k_*, t, A) \text{ for some } k_* > C_2\kappa t \right)
$$
$$
= \sum_{j=1}^\infty P_0\left(X^\kappa \in \Pi(k_*, t, A) \text{ for some } k_* \in (jC_2\kappa t, (j+1)C_2\kappa t] \right).
$$
(3.13)

For $j \in \mathbb{N}$, write $\sum_{(l_1^j, \ldots, l_{t/A}^j)}$ to denote the sum over all sequences $(l_1^j, \ldots, l_{t/A}^j) \in \mathbb{N}^{t/A}$ with $jC_2\kappa t < \sum_{i=1}^{t/A} l_i^j \le (j+1)C_2\kappa t$. Then each summand in (3.13) may, by an application of the Markov property, be rewritten as

$$
\sum_{(l_1^j, \ldots, l_{t/A}^j)} E_0\left(\mathbb{1}\{X^\kappa \text{ has label } (l_1^j, \ldots, l_{t/A-1}^j)\} P_{X^\kappa(t-A)}\left(X^\kappa \text{ has label } l_{t/A}^j \right) \right).
$$
(3.14)

Note that the number of jumps of a path Φ that visits l_i^j 1-blocks is at least $(l_i^j/2^d - 1)A$. This is because for each 1-block there are $(2^d - 1)$ 1-blocks with the same time coordinate at l^∞-distance one. Hence, we may estimate (3.14) from above by

$$
\sum_{(l_1^j, \ldots, l_{t/A}^j)} E_0\left(\mathbb{1}\{X^\kappa \text{ has label } (l_1^j, \ldots, l_{t/A-1}^j)\} \right) P_0\left(N(X^\kappa, A) \ge (l_i^j/2^d - 1)A \right),
$$
(3.15)

where $N(X^\kappa, A)$ denotes the number of jumps of X^κ in the time interval $[0, A)$. An iteration of the arguments in (3.14–3.15), together with the tail estimate $P(\text{POISSON}(\lambda) \ge k) \le e^{-\lambda}(\lambda e)^k/k^k$, $k > 2\lambda + 1$, for Poisson random variables with mean λ, yields that for $C' > 0$ large enough each summand in (3.13) is bounded from above by

$$
\sum_{(l_1^j, \ldots, l_{t/A}^j)} \prod_{l_i^j \ge \kappa C'} P_0\left(N(X^\kappa, A) \ge (l_i^j/2^d - 1)A \right)
$$
$$
\le \sum_{(l_1^j, \ldots, l_{t/A}^j)} \prod_{l_i^j \ge \kappa C'} e^{-A2d\kappa} \exp\left\{ -(l_i^j/2^d - 2)A \log([\kappa C'/2^d - 2]/2d\kappa e) \right\}.
$$
(3.16)

(It suffices to pick C' such that $(C'/2^d - 2)A \ge 4eAd\kappa + 1$, which for $A > 1$ and $\kappa > 1$ is fulfilled when $C' \ge 2^d(4de + 3)$.) Now note that if $C_2 > 2C'$, then for all $j \in \mathbb{N}$,

$$
\sum_{l_i^j \ge \kappa C'} l_i^j \ge \frac{jtC_2\kappa}{2}.
$$
(3.17)

Hence, inserting (3.17) into (3.16), choosing C_2 large enough, and using the fact that for some $a, b \in (0, \infty)$ there are no more than $ae^{b\sqrt{C_2\kappa t}}$ such sequences $(l_1^j, \ldots, l_{t/A}^j)$ (see [7] or [2]), we get that for some $C'' > 0$ the left-hand side of (3.13) is bounded from above by $e^{-C''\kappa t}$. Inserting this bound into (3.11), using that $C'' \to \infty$ as $C_2 \to \infty$, and using (3.12), we get the claim. ∎

4 Proof of Proposition 2.2

Proposition 2.2 is proved in Section 4.2 subject to Propositions 4.1–4.2 below, which are stated in Section 4.1 and proved in Sections 5–6.

4.1 Two more propositions

Endow $\mathbb{Z}$ with the ordering $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec \cdots$. We say that two functions $f, g \colon \mathbb{Z} \to \mathbb{R}$ are equimeasurable when

$$|\{x \in \mathbb{Z} \colon f(x) > \lambda\}| = |\{x \in \mathbb{Z} \colon g(x) > \lambda\}| \qquad \forall\, \lambda \geq 0. \tag{4.1}$$

The *symmetric decreasing rearrangement* of a function $f \colon \mathbb{Z} \to \mathbb{R}$ is defined to be the unique non-increasing function $f^{\sharp} \colon \mathbb{Z} \to \mathbb{R}$ that is equimeasurable with f. Given $A \subseteq \mathbb{Z}$, $A^{\sharp} \subseteq \mathbb{Z}$ is defined to be the unique set such that $(\mathbb{1}_A)^{\sharp} = \mathbb{1}_{A^{\sharp}}$.

For $B \subseteq \mathbb{Z}^d \times [0, t]$, let $\pi_1(B) \subseteq \mathbb{Z} \times [0, t]$ be the projection of the spatial part of B onto its first spatial coordinate. Its one-dimensional symmetric decreasing rearrangement is the set

$$\pi_1(B)^{\sharp} = \bigcup_{s \in [0,t]} \left(\{x \in \mathbb{Z} \colon (x, s) \in \pi_1(B)\}^{\sharp} \times \{s\} \right). \tag{4.2}$$

For $A \geq 1$ and $R \in \mathbb{N}$, an *R-interval* is a time-interval of the form $[kA^R, (k+1)A^R)$, $0 \leq k < t/A^R$. To make the proof more accessible, we no longer distinguish between badness referring to $\xi \mathbb{1}\{\xi \geq K\}$ and badness referring to ξ. Since both potentials satisfy the same mixing assumption (a1) it will be clear from the proof that this does not affect the result.

Proposition 4.1. *Let* $\Phi \in \Pi(k_*, t, A)$. *Then for all* A *large enough there is a sequence* $(\delta_R)_{R \in \mathbb{N}}$ *in* $(0, \infty)$ *satisfying*

$$\sum_{R \in \mathbb{N}} A^{Rd} \sqrt{\delta_R} < \infty \tag{4.3}$$

such that ξ*-a.s. the number of R-intervals in which* Φ *crosses more than* $\delta_R k_*/(t/A)$ *bad R-blocks is bounded from above by* $\sqrt{\delta_R} t/A^R$. *A possible choice is* $\delta_R = K_1 A^{-8d^2/3} A^{-4d(2d+1)R/3}$ *for some* $K_1 > 0$ *not depending on* A *and* R.

Proposition 4.2. *For every* $\varepsilon, t > 0$, *every sequence* $(B_R)_{R \in \mathbb{N}}$ *in* $\mathbb{Z}^d \times [0, t]$ *and every sequence* $(C_R)_{R \in \mathbb{N}}$ *in* $[0, \infty)$ *(see Fig. 3),*

$$E_0\left(\exp\left\{ \sum_{R=1}^{\varepsilon \log t} C_R l_t(B_R) \right\} \right) \leq E_0\left(\exp\left\{ \sum_{R=1}^{\varepsilon \log t} C_R l_t(\pi_1(B_R)^{\sharp}) \right\} \right). \tag{4.4}$$

4.2 Proof of Proposition 2.2 subject to two propositions

Proof. Fix $\varepsilon > 0$ and $A \geq 1$ according to Propositions 2.1 and 4.1, and fix $\tilde{\varepsilon} > 0$. The proof comes in 6 steps.

1. We begin by introducing some more notation. Define the space-time blocks

$$B_R^A(x, k; \kappa) = \left(\prod_{j=1}^d [\sqrt{\kappa}(x(j) - 1)A^R, \sqrt{\kappa}(x(j) + 1)A^R) \cap \mathbb{Z}^d \right) \times [kA^R, (k+1)A^R), \tag{4.5}$$
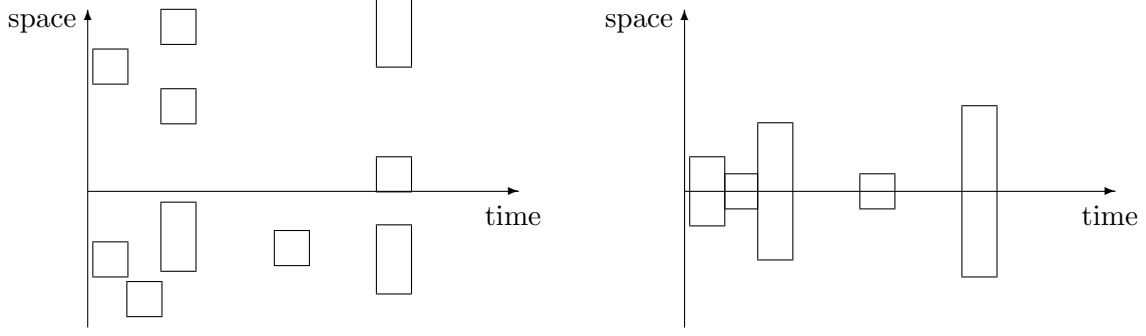
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Figure 3: The picture on the left shows a configuration of space-time blocks before its rearrangement, the picture on the right after its rearrangement. Note that in each time-interval the total space volume of the blocks is the same in both configurations.

which we call (κ, R)-blocks. These blocks are the same as $\tilde{B}_R^{A,\alpha}(x, k; 0, 0) = B_R^{A,\alpha}(x, k)$ in (1.13) with $\alpha = \sqrt{\kappa}$. Let $k_*(\kappa)$ denote the number of $(\kappa, 1)$-blocks that are crossed by X^κ. For $k_*(\kappa) \in \mathbb{N}$ and $(x_i, k_i)_{0 \leq i < k_*(\kappa)} \in (\mathbb{Z}^d \times \mathbb{N})^{k_*(\kappa)}$, write

$$\bigcup_{(x_i,k_i)_{0 \leq i < k_*(\kappa)}} B_1^A(x_i, k_i; \kappa) \tag{4.6}$$

to denote the union over all the $(\kappa, 1)$-blocks $B_1^A(x_i, k_i; \kappa)$, $0 \leq i < k_*(\kappa)$. Likewise, write

$$\sum_{(B_1^A(x_i,k_i;\kappa))_{0 \leq i < k_*(\kappa)}} \tag{4.7}$$

to denote the sum over all possible sequences of $(\kappa, 1)$-blocks $B_1^A(x_i, k_i; \kappa)$, $0 \leq i < k_*(\kappa)$ that can be crossed by a path Φ. Finally, define

$$\mathrm{BAD}_R^\delta((x_i, k_i)_{0 \leq i < k_*(\kappa)}) = \left\{ B_R^A(x, k) \colon\ B_R^A(x, k) \text{ is bad and intersects the union in } (4.6) \right\}. \tag{4.8}$$

2. We write $l_t(\mathrm{BAD}_R^\delta)$ for the local time of X^κ in (δ, b, c)-bad R-blocks up to time t, where badness refers both to $\xi \mathbb{1}\{\xi \geq K\}$ and ξ. By (2.5), it is enough to show that for all κ and t large enough,

$$E_0\left(\exp\left\{ 4 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R^\delta) \right\} \mathbb{1}\{\exists\, k_* \leq C_2 \kappa t \colon X^\kappa \in \Pi(k_*, t, A)\} \right) \leq e^{\tilde{\varepsilon} t}. \tag{4.9}$$

Recall (2.2) to note that the left-hand side of (4.9) equals

$$\sum_{k_*=t/A}^{C_2 \kappa t} E_0\left(\exp\left\{ 4 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R^\delta) \right\} \mathbb{1}\{X^\kappa \text{ crosses } k_* \text{ 1-blocks}\} \right). \tag{4.10}$$

To prove (4.9), we attempt to apply Proposition 4.2. To that end, for each k_* we must sum over all configurations of k_* 1-blocks that may be crossed by X^κ. However, this sum is difficult

to control, and therefore we do an additional *coarse-graining of space-time* by considering (κ, R)-blocks instead of R-blocks. To that end we first note that there is a $C_4 > 0$ such that $k_*(\kappa) \leq C_4 k_*/\sqrt{\kappa} + 2t/A$ (see Lemma 5.6 in Section 5.4 for a similar statement). To see why, note that if Φ crosses $l_i \leq \sqrt{\kappa}$ 1-blocks in the time-interval $[(i-1)A, iA)$, $1 \leq i \leq t/A$, then it crosses $l_i^\kappa \leq 2$ $(\kappa, 1)$-blocks in the same time-interval. Moreover, if $j\sqrt{\kappa} + 1 \leq l_i \leq (j+1)\sqrt{\kappa}$ for some $j \in \mathbb{N}$, then $l_i^\kappa \leq j + 2 \leq (j+2)l_i/j\sqrt{\kappa}$. Hence, the total number of $(\kappa, 1)$-blocks that may be crossed by Φ is bounded from above by

$$\sum_{i=1}^{t/A} l_i^\kappa \leq \sum_{\substack{1 \leq i \leq t/A \\ l_i \leq \sqrt{\kappa}}} 2 + \sum_{\substack{1 \leq i \leq t/A \\ l_i \geq \sqrt{\kappa}+1}} \frac{3l_i}{\sqrt{\kappa}} \leq 2t/A + \frac{3k_*}{\sqrt{\kappa}}. \tag{4.11}$$

Thus, (4.10) is bounded from above by

$$\sum_{k_*(\kappa)=t/A}^{C_2 C_4 \sqrt{\kappa} t + 2t/A} E_0 \left(\exp\left\{ 4 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\big(\mathrm{BAD}_R^\delta\big) \right\} \mathbb{1}\big\{ X^\kappa \text{ crosses } k_*(\kappa) \ (\kappa, 1)\text{-blocks}\big\} \right). \tag{4.12}$$

To analyze (4.12), fix $k_*(\kappa) \in [t/A, C_2 C_4 \sqrt{\kappa} t + 2t/A]$. Summing over all possible ways to cross $k_*(\kappa)$ $(\kappa, 1)$-blocks and recalling (4.8), we may estimate each summand in (4.12) by

$$\sum_{(B_1^A(x_i,k_i;\kappa))_{0 \leq i < k_*(\kappa)}} E_0 \Bigg(\exp\left\{ 4 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\Big(\mathrm{BAD}_R^\delta\big((x_i, k_i)_{0 \leq i < k_*(\kappa)}\big)\Big) \right\} $$
$$\times \mathbb{1}\big\{ X^\kappa \text{ crosses } B_1^A(x_i, k_i; \kappa), 0 \leq i < k_*(\kappa) \big\} \Bigg). \tag{4.13}$$

By Cauchy-Schwarz, (4.13) is at most

$$\sum_{(B_1^A(x_i,k_i;\kappa))_{0 \leq i < k_*(\kappa)}} \left[E_0 \left(\exp\left\{ 8 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\Big(\mathrm{BAD}_R^\delta\big((x_i, k_i)_{0 \leq i < k_*(\kappa)}\big)\Big) \right\} \right) \right]^{1/2}$$
$$\times \left[P_0\Big(X^\kappa \text{ crosses } B_1^A(x_i, k_i, \kappa), 0 \leq i < k_*(\kappa) \Big) \right]^{1/2}. \tag{4.14}$$

3. By Proposition 4.2, the first factor in the summand of (4.14) is not more than

$$\left[E_0 \left(\exp\left\{ 8 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\Big(\pi_1\big(\mathrm{BAD}_R^\delta((x_i, k_i)_{0 \leq i < k_*(\kappa)})\big)^\sharp\Big) \right\} \right) \right]^{1/2}. \tag{4.15}$$

Next, if X^κ crosses $k_*(\kappa)$ $(\kappa, 1)$-blocks $B_1^A(x, k; \kappa)$, then a trivial counting estimate yields that X^κ crosses at most $k_*(\kappa)\sqrt{\kappa}$ 1-blocks. Therefore, by Proposition 4.1, the number of R-intervals in which X^κ crosses more than $\delta_R k_*(\kappa)\sqrt{\kappa}A/t$ bad R-blocks is bounded from above by $\sqrt{\delta_R} t/A^R$. We call these R-intervals R-atypical. Similarly, an R-interval is called R-typical, if the number of bad R-blocks crossed by X^κ is bounded by $\delta_R k_*(\kappa)\sqrt{\kappa}A/t$. Define

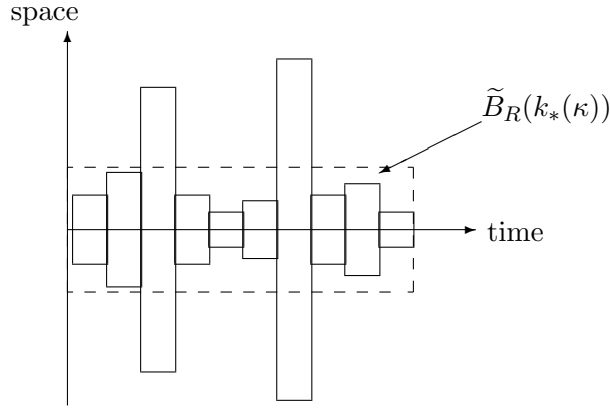$$R^*(k_*(\kappa)) = \max\big\{ R \in \mathbb{N}: \ \delta_R k_*(\kappa)\sqrt{\kappa}A/t \geq 1 \big\}. \tag{4.16}$$

Figure 4: The picture shows a possible configuration of bad R-blocks after its rearrangement. There are two time-intervals in which the number of bad R-blocks is atypically large, i.e., larger than $\delta_R k_*(\kappa)\sqrt{\kappa}A/t$. The local time in these bad R-blocks can be bounded from above by the total length of these time-intervals, which is at most $\sqrt{\delta_R}t$. The local time of the bad R-blocks in the other time-intervals can be bounded from above by the local time of the enveloping dashed block, i.e., $\widetilde{B}_R(k_*(\kappa))$.

If $R > R^*(k_*(\kappa))$, then there are no bad R-blocks in R-typical intervals. (By the choice of R, their number is strictly less than one and therefore is zero.) Hence the local time in bad R-blocks is determined by the local time in bad R-blocks, which lie in R-atypical intervals. Consequently,

$$l_t\Big(\pi_1\big(\mathrm{BAD}_R^\delta((x_i,k_i)_{0\leq i<k_*(\kappa)})\big)^\sharp\Big) \leq (\sqrt{\delta_R}t/A^R)A^R = \sqrt{\delta_R}t. \tag{4.17}$$

On the other hand, if $1 \leq R \leq R^*(k_*(\kappa))$ (see Fig. 4), then there is a contribution coming from R-typical intervals as well, and so

$$l_t\Big(\pi_1\big(\mathrm{BAD}_R^\delta((x_i,k_i)_{0\leq i<k_*(\kappa)})\big)^\sharp\Big) \leq \sqrt{\delta_R}t + l_t(\widetilde{B}_R(k_*(\kappa))), \tag{4.18}$$

where

$$\widetilde{B}_R(k_*(\kappa)) = \Big(\Big[-\tfrac{1}{2}A^R\delta_R k_*(\kappa)\sqrt{\kappa}A/t, \tfrac{1}{2}A^R\delta_R k_*(\kappa)\sqrt{\kappa}A/t\Big) \cap \mathbb{Z}\Big) \times [0,t]. \tag{4.19}$$

Hence, (4.15) is bounded from above by

$$\Bigg[E_0\Bigg(\exp\Bigg\{8\sum_{R=1}^{R^*(k_*(\kappa))}\delta A^{(R+1)d}l_t\big(\widetilde{B}_R(k_*(\kappa))\big)\Bigg\}\Bigg)\Bigg]^{1/2}\exp\Bigg\{4\sum_{R\in\mathbb{N}}\delta A^{(R+1)d}\sqrt{\delta_R}t\Bigg\}. \tag{4.20}$$

For A large enough, by Proposition 4.1 and the specific choice of $(\delta_R)_{R\in\mathbb{N}}$ in Propostion 4.1, the sum in the second term is $\leq \tilde{\varepsilon}t/2$.

4. To estimate the first factor in (4.20) and control the second factor in the summand of (4.14), we need the following two lemmas whose proof is deferred to Appendix A.

Lemma 4.3. *Let X^κ be simple random walk on $\mathbb{Z}$ with step rate 2κ. There is a $K_2 > 0$ such that for all $\kappa > 0$, all $n \in \mathbb{N}$, all $\beta_1, \beta_2, \ldots, \beta_n \geq 0$ and all nested finite intervals $\emptyset = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \mathbb{Z}$,*

$$\log E_0\Bigg(\exp\Bigg\{\sum_{i=1}^n \beta_i \sum_{x\in I_i} l_t(X^\kappa, x)\Bigg\}\Bigg) \leq \frac{K_2 t}{\sqrt{\kappa}}\sum_{i=1}^n\Bigg[|I_i \setminus I_{i-1}|\Bigg(\sum_{j=i}^n \beta_j\Bigg)^{3/2}\Bigg] + o(t), \tag{4.21}$$

where $l_t(X^\kappa, x)$ is the local time of X^κ at site x up to time t.

Lemma 4.4. *There are $C_5, C_6 > 0$ such that for all $\kappa, t > 0$ large enough, all $A > 0$ and all $k_*(\kappa) \geq C_5 t$,*

$$P_0\Big(X^\kappa \text{ crosses } k_*(\kappa) \ (\kappa, 1)\text{-blocks}\Big) \leq e^{-C_6 A k_*(\kappa)}. \tag{4.22}$$

Note that $A^{R+1}\delta_{R+1} < A^R \delta_R$, and so $\widetilde{B}_{R+1}(k_*(\kappa)) \subseteq \widetilde{B}_R(k_*(\kappa))$ for all $R \in \mathbb{N}$. Moreover, for $k_*(\kappa) \leq C_2 C_4 \sqrt{\kappa} t + 2t/A$ and $1 \leq R \leq R^*(k_*(\kappa))$ we have that the cardinality of the spatial part of the blocks defined in (4.19) satisfies $|\widetilde{B}_R(k_*(\kappa))| \leq |\widetilde{B}_1(k_*(\kappa))| \leq A^2 \delta_1 C_2 C_4 \kappa + 2A\delta_1 \sqrt{\kappa}$, which is *bounded uniformly in t*. To apply Lemma 4.3, we choose t_0 (which may depend on κ) such that for each family of intervals $I_1, \ldots, I_{R^*(k_*)}$, $k_*(\kappa) \in [t/A, C_2 C_4 \sqrt{\kappa} t + 2t/A]$, with the property that $|I_i| \in [A, A^2 \delta_1 C_2 C_4 \kappa + 2A\delta_1 \sqrt{\kappa}]$ for all $i \in \{1, \ldots, R^*(k_*(\kappa))\}$ the assertion of Lemma 4.3 holds uniformly in $t \geq t_0$. Then, for all $t \geq t_0$, the expectation in the left-hand side of (4.20) is at most

$$\exp\left\{\frac{K_2 t}{\sqrt{\kappa}} \sum_{R=1}^{R^*(k_*(\kappa))} \left[\Big|\widetilde{B}_R(k_*(\kappa)) \setminus \widetilde{B}_{R+1}(k_*(\kappa))\Big|\left(\sum_{j=1}^{R} 8\delta A^{(j+1)d}\right)^{3/2}\right] + o(t)\right\}, \tag{4.23}$$

where $\widetilde{B}_{R^*(k_*(\kappa))+1}(k_*(\kappa)) = \emptyset$. Next, note that

$$|\widetilde{B}_R(k_*(\kappa)) \setminus \widetilde{B}_{R+1}(k_*(\kappa))| \leq \frac{A^R \delta_R k_*(\kappa)\sqrt{\kappa} A}{t}. \tag{4.24}$$

Therefore the first term in the exponent of (4.23), may be estimated from above by

$$\frac{K_2 t}{\sqrt{\kappa}} \sum_{R=1}^{R^*(k_*(\kappa))} \left[\frac{A^R \delta_R k_*(\kappa)\sqrt{\kappa} A}{t}\left(\sum_{j=1}^{R} 8\delta A^{(j+1)d}\right)^{3/2}\right]$$

$$\leq (8\delta)^{3/2} K_2 k_*(\kappa) A^{3d/2+1} \sum_{R=1}^{R^*(k_*(\kappa))} \left[A^R \delta_R \left(\sum_{j=1}^{R} A^{jd}\right)^{3/2}\right]. \tag{4.25}$$

Furthermore,

$$\sum_{j=1}^{R} A^{jd} = \frac{A^d}{A^d - 1}(A^{Rd} - 1) \leq C A^{Rd}, \tag{4.26}$$

where $C > 0$ does not depend on A. Hence, the right-hand side of (4.25) is at most

$$(8\delta)^{3/2} C K_2 k_*(\kappa) A^{3d/2+1} \sum_{R=1}^{R^*(k_*(\kappa))} A^R \delta_R A^{3Rd/2}. \tag{4.27}$$

Recalling our choice of δ_R in Proposition 4.1, we can estimate the sum in (4.27) from above by

$$K_1 A^{-8d^2/2} A^{-D(d)}\left[\frac{1 - A^{-R^*(k_*(\kappa))D(d)}}{1 - A^{-D(d)}}\right]. \tag{4.28}$$

with $D(d) = (16d^2 - d - 6)/6 > 0$. Since $A > 3$ by Proposition 2.1, the last term in (4.28) is bounded uniformly in A and $R^*(k_*(\kappa))$. Inserting (4.28) into (4.27), we see that there is a $C_7 > 0$, not depending on A, such that the exponent in (4.23) is bounded from above by

$(8\delta)^{3/2}C_7 A^{-D'(d)}k_*(\kappa) + o(t)$ with $D'(d) = (16d^2 - 5d - 6)/3 > 0$, where $o(t)$ is uniform in $t \geq t_0$ for all $k_*(\kappa) \in [t/A, C_2C_4\sqrt{\kappa}t + 2t/A]$.

5. It remains to estimate (recall (4.14))

$$\sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} \left[P_0\Big(X^\kappa \text{ crosses } B_1^A(x_i, k_i, \kappa), 0 \leq i < k_*(\kappa)\Big) \right]^{1/2}. \qquad (4.29)$$

Let $\left| \sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} \right|$ denote the cardinality of the sum in (4.29). By Jensen's inequality, (4.29) is not more than (recall (4.7))

$$\left| \sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} \right|^{1/2} \left[\sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} P_0\Big(X^\kappa \text{ crosses } B_1^A(x_i, k_i; \kappa), 0 \leq i < k_*(\kappa)\Big) \right]^{1/2}$$

$$= \left| \sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} \right|^{1/2} \left[P_0\Big(X^\kappa \text{ crosses } k_*(\kappa) \ (\kappa, 1)\text{-blocks}\Big) \right]^{1/2}.$$

$$(4.30)$$

To estimate the first term in the right-hand side of (4.30), note that $\left| \sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} \right|$ equals the number of different ways to visit $k_*(\kappa)$ $(\kappa, 1)$-blocks. Hence, there is a $C_8 > 0$ such that $\left| \sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i < k_*(\kappa)}} \right|$ is bounded from above by $e^{C_8 k_*(\kappa)}$ (see also Lemma 5.5 in Section 5.4). Therefore, by Lemma 4.4, for $k_*(\kappa) \geq C_5 t$ and κ large enough, the right-hand side of (4.30) may be estimated from above by

$$e^{C_8 k_*(\kappa)} e^{-C_6 A k_*(\kappa)}. \qquad (4.31)$$

6. We are now in a position to complete the proof of (4.9). Combining the estimates in (4.15), (4.20) and (4.25–4.31), we get for $t \geq t_0$ (see the lines following (4.28)),

$$E_0\left(\exp\left\{ 4 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R) \right\} \mathbb{1}\big\{ \exists\, k_* \leq C_2\kappa t \colon X^\kappa \in \Pi(k_*, t, A) \big\} \right)$$

$$\leq e^{\tilde\varepsilon t/2} \sum_{k_*(\kappa)=t/A}^{C_5 t - 1} e^{(8\delta)^{3/2}C_7 A^{-D'(d)}k_*(\kappa)+o(t)}$$

$$\qquad\qquad + e^{\tilde\varepsilon t/2} \sum_{k_*(\kappa)=C_5 t}^{C_2 C_4\sqrt{\kappa}t+2t/A} e^{(8\delta)^{3/2}C_7 A^{-D'(d)}k_*(\kappa)+o(t)} e^{C_8 k_*(\kappa)} e^{-C_6 A k_*(\kappa)} \qquad (4.32)$$

$$\leq e^{\tilde\varepsilon t/2} C_5 t \, e^{(8\delta)^{3/2}C_7 C_5 A^{-D'(d)}t+o(t)} + e^{\tilde\varepsilon t/2} C_9$$

$$\leq e^{2\tilde\varepsilon t},$$

where we use that the sum in the third line of (4.32) is finite for A large enough (which requires that ε is small enough; recall Proposition 2.1). This settles (4.9) and completes the proof of Proposition 2.2. ∎

5 Proof of Proposition 4.1

The proof is given in Section 5.1 subject to Lemma 5.1 below. This lemma is stated in Section 5.1 and proved in Sections 5.2–5.5. Recall the definition of Ξ_R^{A,k_*} in (3.1). Throughout this section we abbreviate

$$\widetilde{\delta}_R = A^{-2d(2d+1)R}. \tag{5.1}$$

5.1 Proof of Proposition 4.1 subject to a further lemma

Lemma 5.1. *There is a $C > 0$ such that ξ-a.s. for all A and m large enough, all $R \in \mathbb{N}$ and all $k_* \in \mathbb{N}$,*

$$\Xi_R^{A,k_*} \le C A^{-(4d^2-1)} A^{-R} \widetilde{\delta}_R k_*. \tag{5.2}$$

We are now ready to prove Proposition 4.1.

Proof. Let $\Phi \in \Pi(k_*, t, A)$ and $R \in \mathbb{N}$. Suppose that there is a $\delta_R > 0$ such that there are at least $\sqrt{\delta_R} t / A^R$ R-intervals in which Φ crosses more than $\delta_R k_*/(t/A)$ bad R-blocks. In all of these R-intervals Φ crosses at least

$$\frac{\sqrt{\delta_R} t}{A^R} \frac{\delta_R k_*}{(t/A)} = \delta_R^{3/2} A^{-(R-1)} k_* \tag{5.3}$$

bad R-blocks. Lemma 5.1 implies that ξ-a.s. $\delta_R^{3/2} A^{-(R-1)} \le C A^{-(4d^2-1)} A^{-R} \widetilde{\delta}_R$, which is the same as $\delta_R^{3/2} \le C A^{-4d^2} A^{-2d(2d+1)R}$. This yields the claim below (4.3) with $K_1 = C^{2/3}$. ∎

5.2 Proof of Lemma 5.1 subject to two further lemmas

The proof of Lemma 5.1 is a modification of the proof of [3, Lemma 3.5] and is based on Lemmas 5.2–5.3 below, which count bad R-blocks. The proof of the second lemma is deferred to Section 5.3.

For $A \ge 1$, $R \in \mathbb{N}$ and $\Phi \in \Pi(k_*, t, A)$, define

$$\Psi_R^A(\Phi) = \text{number of good } (R+1)\text{-blocks crossed by } \Phi \text{ containing a bad } R\text{-block},$$
$$\Psi_R^{A,k_*} = \sup_{\Phi \in \Pi(k_*,t,A)} \Psi_R^A(\Phi). \tag{5.4}$$

Lemma 5.2. *There is a $C' > 0$ such that for all A and m large enough*

$$\mathbb{P}\Big(\Psi_R^{A,k_*} \ge C' A^{-R} \widetilde{\delta}_R k_* \text{ for some } R \in \mathbb{N} \text{ and some } k_* \in \mathbb{N}_0 \Big) \tag{5.5}$$

is summable over $t \in \mathbb{N}$. A possible choice is $C' = 3$.

Lemma 5.3. *For all $\varepsilon > 0$ there is an $A = A(\varepsilon) > 3$ such that ξ-a.s. there is a $t_0 > 0$ such that for all $R \in \mathbb{N}$, all $k_* \in \mathbb{N}$ and all $t \ge t_0$,*

$$\Xi_R^{A,k_*} \le A^{d+1} \sum_{i=1}^{\varepsilon \log t - R - 1} 2^{di} A^{(d+1)i} \Psi_{R+i}^{A,k_*}. \tag{5.6}$$

Proof. Lemma 5.3 is the same as [3, Lemma 3.7]. The idea is to look at a bad R-block and check whether it is contained in a good $(R+1)$-block or in a bad $(R+1)$-block. An iteration over R, combined with a simple counting argument and Lemma 3.1, yields the claim. ∎

We are now ready to prove Lemma 5.1.

Proof. By Lemma 5.2, ξ-a.s. for t large enough $\Psi_R^{A,k_*} \leq C' A^{-R} \widetilde{\delta}_R k_*$ for all $R \in \mathbb{N}$ and all $k_* \in \mathbb{N}$. By Lemma 5.3, recalling that $\widetilde{\delta}_R = A^{-2d(2d+1)R}$, we may estimate

$$
\begin{aligned}
\Xi_R^{A,k_*} &\leq A^{d+1} \sum_{i \geq 1} 2^{di} A^{(d+1)i} C' A^{-(R+i)} \widetilde{\delta}_{R+i} k_* \\
&= C' A^{d+1} A^{-R} \widetilde{\delta}_R k_* \sum_{i \geq 1} 2^{di} A^{(d+1)i} A^{-i} A^{-2d(2d+1)i} \\
&= C' A^{d+1} A^{-R} \widetilde{\delta}_R k_* \frac{2^d A^d A^{-2d(2d+1)}}{1 - 2^d A^d A^{-2d(2d+1)}}.
\end{aligned}
\tag{5.7}
$$

Note that for $A \geq A_0 > 1$ there is a $C > 0$, depending on A_0 but not on A, such that the term in the right-hand side of (5.7) is bounded from above by

$$
C A^{-(4d^2-1)} A^{-R} \widetilde{\delta}_R k_*,
\tag{5.8}
$$

which yields the claim. ∎

5.3 Proof of Lemma 5.2 subject to a further lemma

The proof of Lemma 5.2 is based on Lemma 5.4 below. Let $x \in \mathbb{Z}^d$ and $k, R \in \mathbb{N}$. Abbreviate

$$
\chi^A(x,k) = \mathbb{1}\{B_{R+1}^A(x,k) \text{ is good but contains a bad } R\text{-block}\}.
\tag{5.9}
$$

Lemma 5.4. *There is a $C > 0$ such that for all A and m large enough, all $R \in \mathbb{N}$ and all $k_* \in \mathbb{N}$,*

$$
\mathbb{P}\left(\begin{array}{c} \text{there is a path that crosses } k_* \text{ 1-blocks and intersects} \\ \text{at least } 3A^{-R}\widetilde{\delta}_R k_* \text{ blocks } B_{R+1}^A(x,k) \text{ with } \chi^A(x,k)=1 \end{array}\right) \leq \exp\left\{-CA^{-R}\widetilde{\delta}_R k_*\right\}.
\tag{5.10}
$$

We are now ready to prove Lemma 5.2.

Proof. First note that $k_* \geq t/A$ and that, ξ-a.s. for t large enough, $1 \leq R \leq \varepsilon \log t$, by Lemma 3.1. For each such R, we have by Lemma 5.4,

$$
\begin{aligned}
&\mathbb{P}\left(\begin{array}{c} \text{there is a path that crosses } k_* \text{ 1-blocks and intersects at least} \\ 3A^{-R}\widetilde{\delta}_R k_* \text{ blocks } B_{R+1}^A(x,k) \text{ with } \chi^A(x,k)=1 \text{ for some } k_* \geq t/A \end{array}\right) \\
&\leq \sum_{k_* \geq t/A} \exp\{-CA^{-R}\widetilde{\delta}_R k_*\} \leq \frac{\exp\{-CA^{-R}\widetilde{\delta}_R t/A\}}{1 - \exp\{-CA^{-R}\widetilde{\delta}_R\}}.
\end{aligned}
\tag{5.11}
$$

Because $1 \leq R \leq \varepsilon \log t$ and $R \mapsto A^{-R}\widetilde{\delta}_R$ is non-increasing, the numerator in the right-hand side of (5.11) is bounded from above by $\exp\{-CA^{-\varepsilon \log t}\widetilde{\delta}_{\varepsilon \log t} t/A\}$ while the denominator is

bounded from below by $1 - \exp\{-CA^{-\varepsilon \log t}\widetilde{\delta}_{\varepsilon \log t}\}$. Using the choice of A in Lemma 3.1, we see that (5.11) is bounded from above by

$$\frac{\exp\{-Ct^{1-a^{-1}}/A\}}{1 - \exp\{-Ct^{-a^{-1}}\}}, \qquad a > 1. \tag{5.12}$$

Note that this is of order $\exp\{-C't^{\tilde{\varepsilon}}\}$ for some $C', \tilde{\varepsilon} > 0$, and so the probability in (5.5) is bounded from above by $(\varepsilon \log t) \exp\{-C't^{\tilde{\varepsilon}}\}$, which is summable over $t \in \mathbb{N}$. ∎

5.4 Proof of Lemma 5.4 subject to two further lemmas

The proof of Lemma 5.4 is based on Lemmas 5.5–5.6 below, which are proved in Section 5.5.

Proof. Our first further lemma reads:

Lemma 5.5. *There is a $C > 0$ such that for all $l \in \mathbb{N}$ and $R \in \mathbb{N}$ there are no more than e^{Cl} possible ways for Φ to visit at most l R-blocks.*

Fix $R \in \mathbb{N}$. We divide blocks into equivalence classes such that blocks belonging to the same equivalence class can essentially be treated as independent. To that end, we take $a_1, a_2 \in \mathbb{N}$ according to condition (a1) in Definition 1.5 and say that (x, k) and (x', k') are equivalent when

$$x = x' \,(\mathrm{mod}\, a_1), \qquad k = k' \,(\mathrm{mod}\, a_2). \tag{5.13}$$

We denote the set of corresponding representatives by $([x], [k])$, and write $\sum_{([x],[k])}$ to denote the sum over all equivalence classes. Note that the left-hand side of (5.10) is bounded from above by

$$\sum_{([x],[k])} \mathbb{P}\left(\begin{array}{l} \text{there is a path that crosses } k_* \text{ 1-blocks and intersects} \\ \text{at least } 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2 \text{ blocks } B_{R+1}^A(x,k) \\ \text{with } \chi^A(x,k) = 1 \text{ and } (x,k) \equiv ([x],[k]) \end{array}\right). \tag{5.14}$$

Fix an equivalence class. Put $\rho_R = A^{-4d(2d+1)(d+1)R}$ (recall (1.17)). To control the cardinality of the number of different ways to visit a given number of $(R+1)$-blocks, we consider enlarged blocks, namely, we let

$$L = L(R) = (1/\rho_R)^{1/(d+1)} \tag{5.15}$$

and define

$$\tilde{B}_R^A(x,k) = \left(\prod_{j=1}^{d} \left[Lx(j)A^R, L(x(j)+1)A^R\right) \cap \mathbb{Z}^d\right) \times [LkA^R, L(k+1)A^R). \tag{5.16}$$

Our second further lemma reads:

Lemma 5.6. *If Φ crosses k_* 1-blocks, then for all $R \in \mathbb{N}$ it crosses no more than $l_R = 3k_*/A^{R-1}L$ blocks $\widetilde{B}_R^A(x,k)$.*

We write

$$\bigcup_{(x_i,k_i)_{0 \le i < l_{R+1}}} \tilde{B}_{R+1}^A(x_i, k_i) \tag{5.17}$$

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to denote the union over at most l_{R+1} blocks $\widetilde{B}_R^A(x,k)$ and

$$\sum_{(\tilde{B}_{R+1}^A(x_i,k_i))_{0\le i<l_{R+1}}} \tag{5.18}$$

to denote the sum over all possible sequences of at most l_{R+1} blocks $\tilde{B}_{R+1}^A(x_i,k_i)$ that can be crossed by a path Φ. Since each block $B_{R+1}^A(x,k)$ that may be crossed by Φ lies in the union of (5.17), we may estimate the probability under the sum in (5.14) from above by

$$\sum_{(\tilde{B}_{R+1}^A(x_i,k_i))_{0\le i<l_{R+1}}} \mathbb{P}\left(\begin{array}{l}\text{the union in (5.17) contains at least } 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2 \text{ blocks} \\ B_{R+1}^A(x,k) \text{ with } \chi^A(x,k)=1 \text{ and } (x,k)\equiv([x],[k])\end{array}\right).$$

$$\tag{5.19}$$

Next, note that the union in (5.17) contains at most $l_{R+1}L^{d+1}$ $(R+1)$-blocks and that there are $\binom{l_{R+1}L^{d+1}}{n}$ ways of choosing n blocks $B_{R+1}^A(x,k)$ with $\chi^A(x,k)=1$ from $l_{R+1}L^{d+1}$ $(R+1)$-blocks. Hence, by the mixing condition in (1.17) for A and m large enough, each summand in (5.19) is bounded from above by

$$\sum_{n=\delta_R k_{R+1}/a_1^d a_2}^{l_{R+1}L^{d+1}} \binom{l_{R+1}L^{d+1}}{n}(\rho_R)^n \le (1-\rho_R)^{-l_{R+1}L^{d+1}}\mathbb{P}\left(T\ge 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2\right), \tag{5.20}$$

where $T=\text{BINOMIAL}(l_{R+1}L^{d+1},\rho_R)$. Since

$$\mathbb{E}(T)=\rho_R l_{R+1}L^{d+1}=l_{R+1}=3k_*/A^R L=3A^{-R}A^{-4d(2d+1)R}k_*=3A^{-R}\widetilde{\delta}_R^2 k_* \ll 3A^{-R}\widetilde{\delta}_R k_*, \tag{5.21}$$

we can apply standard large deviation estimates to bound the right-hand side of (5.20). Indeed, by Bernstein's inequality, there is a $C'>0$ (depending on a_1 and a_2 only) such that for all A and m large enough,

$$\mathbb{P}\left(T\ge 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2\right) \le e^{-C'3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2}. \tag{5.22}$$

Moreover, there is a $C''>0$ (not depending on A, provided A is large enough) such that

$$(1-\rho_R)^{-l_{R+1}L^{d+1}} \le e^{\rho_R l_{R+1}L^{d+1}/(1-\rho_R)} \le e^{C''3A^{-R}\widetilde{\delta}_R^2 k_*}. \tag{5.23}$$

Furthermore, by Lemma 5.5, and after a possible increase of C'', the sum in (5.18) contains at most $e^{C''l_{R+1}}=e^{C''3A^{-R}\widetilde{\delta}_R^2 k_*}$ elements. Hence, combining (5.14), (5.19–5.20) and (5.22–5.23), we see that the left-hand side of (5.10) is bounded from above by $e^{-CA^{-R}\widetilde{\delta}_R k_*}$, with C such that $CA^{-R}\widetilde{\delta}_R k_* \ge (C'/a_1^d a_2 - \widetilde{\delta}_R 2C'')3A^{-R}\widetilde{\delta}_R k_*$, which yields the claim in (5.10). $\blacksquare$

5.5 Proof of Lemmas 5.5–5.6

Proof. For the proof of Lemma 5.5, see the proof of [3, Claim 3.8]. The proof of Lemma 5.6 goes as follows. Let $R\in\mathbb{N}$. Divide time into intervals of length LA^R. Let l_i^L and l_i be the number of blocks $\widetilde{B}_R^A(x,k)$, respectively, 1-blocks, crossed by X^κ in the i-th time interval $[(i-1)LA^R,iLA^R)$, $1\le i\le t/LA^R$. Note that $l_i\ge LA^{R-1}$ because the length of the time-interval of each block $\widetilde{B}_R^A(x,k)$ is LA^R, which may be divided into LA^{R-1} time-intervals of

length A. Moreover X^κ has to cross at least one 1-block in each such interval of length A. Also note that if $l_i = LA^{R-1}$, then $l_i^L \leq 2 = 2l_i/l_i \leq 2l_i^1/LA^{R-1}$. If $LA^{R-1}+1 \leq l_i \leq 2LA^{R-1}$, then $l_i^L \leq 3$, because X^κ may start at an interface between two blocks $\widetilde{B}_R^A(x,k)$ and immediately jump from one such block to another. However, to afterwards reach the next block $\widetilde{B}_R^A(x,k)$ it has to cross at least LA^{R-1} 1-blocks, and so $l_i^L \leq 3l_i/l_i \leq 3l_i^1/LA^{R-1}$. Furthermore, for $j \in \mathbb{N}$, if $jLA^{R-1} + 1 \leq l_i \leq (j+1)LA^{R-1}$, then

$$l_i^L \leq (j+2)l_i/l_i \leq (j+2)l_i/jLA^{R-1}. \tag{5.24}$$

Therefore we have

$$k_* = \sum_{i=1}^{t/LA^{R-1}} l_i \geq \frac{LA^{R-1}}{3} \sum_{i=1}^{t/LA^{R-1}} l_i^L, \tag{5.25}$$

or $\sum_{i=1}^{t/LA^{R-1}} l_i^L \leq (3/LA^{R-1})k_* = l_R$, which completes the proof. ■

6 Proof of Proposition 4.2

In Section 6.1 we reduce the problem to one dimension and recall two discrete rearrangement inequalities from the literature (Propositions 6.3–6.4 below). In Section 6.2 we use the latter to give the proof of Proposition 4.2.

6.1 Reduction to one dimension and discrete rearrangement inequalities

Lemma 6.1. *Let $B \subseteq \mathbb{Z}^d \times [0,t]$. Then, for all $C \geq 0$,*

$$E_0\big(e^{Cl_t(B)}\big) \leq E_0\big(e^{Cl_t(\pi_1(B))}\big). \tag{6.1}$$

Proof. A d-dimensional simple random walk with jump rate $2d\kappa$ is a vector of d independent one-dimensional simple random walks, each having jump rate 2κ. Hence, given any set $B \subseteq \mathbb{Z}^d \times [0,t]$,

$$\forall s \geq 0: \qquad X^\kappa(s) \in B \implies \pi_1(X^\kappa)(s) \in \pi_1(B). \tag{6.2}$$

This in turn implies that $l_t(B) \leq l_t(\pi_1(B))$, which proves the claim. ■

To prove Proposition 4.2 we need two discrete rearrangement inequalities [13], [14]. For an overview on continuous rearrangement inequalities we refer the reader to [12, Chapter 3].

Definition 6.2. *A function $L\colon \mathbb{Z} \times \mathbb{Z} \to [0,\infty)$ is called of Riesz-type when, for all pairs of functions $f,g\colon \mathbb{Z} \to [0,\infty)$,*

$$\sum_{x,y \in \mathbb{Z}} f(x)L(x,y)g(y) \leq \sum_{x,y \in \mathbb{Z}} f^\sharp(x)L(x,y)g^\sharp(y). \tag{6.3}$$

Proposition 6.3. *[13, Theorem 2.2], [14]) Let $K\colon [0,\infty) \to [0,\infty)$ be non-increasing. Then $L\colon \mathbb{Z} \times \mathbb{Z} \to [0,\infty)$ given by $L(x,y) = K(|x-y|)$ is of Riesz-type.*

Note that $(x,y) \mapsto p_s^\kappa(x,y)$ with $p_s^\kappa(x,y)$ the transition kernel of one-dimensional simple random walk with jump rate 2κ is of Riesz-type. Indeed, $p_s^\kappa(x,y) = p_s^\kappa(x-y,0) = p_s^\kappa(|x-y|,0)$ is a non-increasing function of $|x-y|$.

The following multiple-sum version of Proposition 6.3 will be needed also.

Proposition 6.4. ([13, Lemma 9.1 in Chapter 2], [14]) *Fix $n \in \mathbb{N}$. Let $L_0, L_1, \ldots, L_{n-1}$ be a collection of Riesz-type functions on $\mathbb{Z} \times \mathbb{Z}$, and let $S_0, S_1, \ldots, S_n$ be a collection of non-negative functions on $\mathbb{Z}$. Then*

$$\sum_{x_0, x_1, \ldots, x_n \in \mathbb{Z}} \left(\prod_{i=0}^{n-1} S_i(x_i) L_i(x_i, x_{i+1}) \right) S_n(x_n) \leq \sum_{x_0, x_1, \ldots, x_n \in \mathbb{Z}} \left(\prod_{i=0}^{n-1} S_i^{\sharp}(x_i) L_i(x_i, x_{i+1}) \right) S_n^{\sharp}(x_n). \tag{6.4}$$

6.2 Proof of Proposition 4.2

Proof. Let $(B_R)_{R \in \mathbb{N}}$ be a sequence in $\mathbb{Z} \times [0, t]$ (recall Lemma 6.1) and $(C_R)_{R \in \mathbb{N}}$ a sequence of nonnegative numbers. Write

$$E_0 \left(\exp \left\{ \sum_{R \in \mathbb{N}} C_R l_t(B_R) \right\} \right) = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} E_0 \left(\left\{ \sum_{R \in \mathbb{N}} C_R l_t(B_R) \right\}^n \right). \tag{6.5}$$

The n-th moments in (6.5) may be rewritten as

$$\sum_{R_1, \ldots, R_n \in \mathbb{N}} \left(\prod_{i=1}^{n} C_{R_i} \right) E_0 \left(\prod_{i=1}^{n} l_t(B_{R_i}) \right). \tag{6.6}$$

Write out

$$\prod_{i=1}^{n} l_t(B_{R_i}) = \int_0^t ds_1 \ldots \int_0^t ds_n \, \mathbb{1}\left\{ X^{\kappa}(s_1) \in B_{R_1}, \ldots X^{\kappa}(s_n) \in B_{R_n} \right\}, \tag{6.7}$$

so that the second factor under the sum in (6.6) equals

$$\int_0^t ds_1 \ldots \int_0^t ds_n \, P_0 \left(X^{\kappa}(s_1) \in B_{R_1}, \ldots X^{\kappa}(s_n) \in B_{R_n} \right). \tag{6.8}$$

Fix a choice of $(s_1, \ldots, s_n) \in [0, t]^n$, and let $B_{R_{s_i}}$ be the spatial part of $B_{R_i} \cap (\mathbb{Z} \times \{s_i\})$. Without loss of generality we may assume that $s_1 < s_2 < \ldots < s_n$, so that the probability in (6.8) becomes ($x_0 = 0$, $s_0 = 0$)

$$\sum_{x_1, \ldots, x_n \in \mathbb{Z}} \left(\prod_{i=0}^{n} \mathbb{1}\left\{ x_i \in B_{R_{s_i}} \right\} \right) \left(\prod_{i=0}^{n-1} p_{s_{i+1} - s_i}^{\kappa}(x_i, x_{i+1}) \right). \tag{6.9}$$

An application of Proposition 6.4 gives that (6.9) is bounded from above by

$$\sum_{x_1, \ldots x_n \in \mathbb{Z}} \left(\prod_{i=0}^{n} \mathbb{1}\left\{ x_i \in B_{R_{s_i}}^{\sharp} \right\} \right) \left(\prod_{i=0}^{n-1} p_{s_{i+1} - s_i}^{\kappa}(x_i, x_{i+1}) \right), \tag{6.10}$$

so that, by (6.8),

$$E_0 \left(\prod_{i=1}^{n} l_t(B_{R_i}) \right) \leq E_0 \left(\prod_{i=1}^{n} l_t(B_{R_i}^{\sharp}) \right). \tag{6.11}$$

Inserting this back into (6.5) and (6.6), we get the claim. ∎

7 Proof of Proposition 2.3

In Section 7.1 we introduce some notation and state two more propositions, Propositions 7.3–7.4 below, whose proof is given in Sections 7.3–7.4. In Section 7.2 we give the proof of Proposition 2.3 subject to these propositions.

7.1 Two more propositions

Henceforth we assume that α in (1.13) takes the form $\alpha = 4M\kappa$ with M a constant that will be determined later on. Recall the definition of π_1 below (2.2) and of $\bar{\xi}$ in (2.4).

Definition 7.1. *The subpedestal of $B_1^{A,4M\kappa}(x,k)$ is (see Fig. 5)*

$$B_{1,\text{sub}}^{A,4M\kappa}(x,k) = \Big\{ y \in \pi_1\big(B_1^{A,4M\kappa}(x,k)\big) :$$
$$|y(j) - z(j)| \geq 2M\kappa A,\, j \in \{1,2,\ldots,d\}\, \forall\, z \in \partial\pi_1\big(B_1^{A,4M\kappa}(x,k)\big) \Big\} \times \{kA\}. \tag{7.1}$$

Definition 7.2. *Let $\varepsilon > 0$, and $k,n \in \mathbb{N}_0$ such that $n \geq k$. A block $B_1^{A,4M\kappa}(x,k)$ is called ε-sufficient at level n when, for every $y \in \pi_1(B_{1,\text{sub}}^{A,4M\kappa}(x,k))$,*

$$E_y\left(\exp\left\{ \int_0^A \bar{\xi}(X^\kappa(s), A(n-k) - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, A) \leq M\kappa A \big\} \right) \leq e^{\varepsilon A}. \tag{7.2}$$

Otherwise it is called ε-insufficient at level n. A subpedestal is called ε-(in)sufficient at level n when its corresponding block is ε-(in)sufficient at level n.

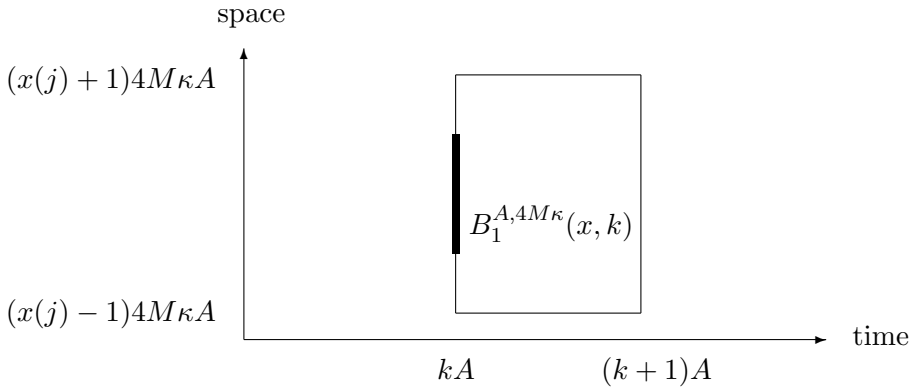
Figure 5: The thick line is the subpedestal.

Proposition 7.3. *Let $A > 1$. There is a constant $C_3 > 0$ such that for all $n \in \mathbb{N}$ the number of different sequences of subpedestals $B_{1,\text{sub}}^{A,4M\kappa}(0,0), B_{1,\text{sub}}^{A,4M\kappa}(x_1,1), \ldots, B_{1,\text{sub}}^{A,4M\kappa}(x_{n-1}, n-1)$ with the property that there is a path $\Phi\colon [0,An] \to \mathbb{Z}^d$ with at most $M\kappa An$ jumps satisfying $\Phi(kA) \in B_{1,\text{sub}}^{A,4M\kappa}(x_k,k)$, $k \in \{0,1,\ldots,n-1\}$, is bounded from above by $e^{C_3 n}$.*

Proposition 7.4. *Fix $\varepsilon > 0$. Let $\delta = \frac{1}{4}\varepsilon$ in the definition of $\bar{\xi}$ and $A > 1$. Then there is a $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$ and ξ-a.s. for all $n \in \mathbb{N}$, all blocks $B_1^{A,4M\kappa}(x,k)$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $k \leq n$, are ε-sufficient at level n.*

7.2 Proof of Proposition 2.3 subject to two propositions

Proof. The proof comes in 2 Steps.

1. Fix $\varepsilon > 0$ and put $\delta = \frac{1}{4}\varepsilon$. Choose $\kappa \geq \kappa_0$ according to Proposition 7.4. Then the tail estimate $P\big(\text{POISSON}(\lambda) \geq k\big) \leq e^{-\lambda}(\lambda e)^k/k^k$, $k \geq 2\lambda + 1$, for Poisson-distributed random variables with mean λ shows that, for $M > 0$ large enough,

$$
\begin{aligned}
&E_0\left(\exp\left\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, An) > M\kappa An \big\} \right) \\
&\qquad \leq e^{2\delta A^{d+1}n} e^{-2d\kappa An} \exp\big\{ -M\kappa An[\log(M/2d) - 1] \big\},
\end{aligned}
\tag{7.3}
$$

where we use (2.4). Since we later let $\kappa \to \infty$, (7.3) shows that it is enough to concentrate on contributions coming from paths with at most $M\kappa An$ jumps. To that end, fix a $\mathbb{Z}^d$-valued sequence of vertices $x_0, x_1, \ldots, x_{n-1}$ such that $x_0 = 0$ and such that there is a path that starts in 0, makes $0 \leq j \leq M\kappa An$ jumps, and is in the subpedestal $B_{1,\text{sub}}^{A,4M\kappa}(x_k, k)$ at time kA for $k \in \{0, 1, \ldots, n-1\}$. By the Markov property of X^κ applied at times kA, $k \in \{1, 2, \ldots, n-1\}$,

$$
\begin{aligned}
&E_0\left(\exp\left\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, An) \leq M\kappa An \big\} \right. \\
&\qquad\qquad \left. \times \prod_{k=1}^{n-1} \mathbb{1}\big\{ X^\kappa(kA) \in B_{1,\text{sub}}^{A,4M\kappa}(x_k, k) \big\} \right) \\
&\qquad \leq \prod_{k=0}^{n-1} \sup_{y \in \pi_1\big(B_{1,\text{sub}}^{A,4M\kappa}(x_k,k)\big)} E_y\left(\exp\left\{ \int_0^A \overline{\xi}(X^\kappa(s), A(n - k) - s)\, ds \right\} \right).
\end{aligned}
\tag{7.4}
$$

This is at most

$$
\begin{aligned}
&\prod_{k=0}^{n-1} \left[\sup_{y \in \pi_1\big(B_{1,\text{sub}}^{A,4M\kappa}(x_k,k)\big)} E_y\left(\exp\left\{ \int_0^A \overline{\xi}(X^\kappa(s), A(n - k) - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, A) \leq M\kappa A \big\} \right) \right.\\
&\qquad\qquad \left. + \sup_{y \in \pi_1\big(B_{1,\text{sub}}^{A,4M\kappa}(x_k,k)\big)} E_y\left(\exp\left\{ \int_0^A \overline{\xi}(X^\kappa(s), A(n - k) - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, A) > M\kappa A \big\} \right) \right] \\
&= \sum_{J \subset \{0,1,\ldots,n-1\}} \Bigg[\\
&\quad \prod_{k \in J} \sup_{y \in \pi_1(B_{1,\text{sub}}^{A,4M\kappa}(x_k,k))} E_y\left(\exp\left\{ \int_0^A \overline{\xi}(X^\kappa(s), A(n - k) - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, A) \leq M\kappa A \big\} \right) \\
&\quad \times \prod_{k \notin J} \sup_{y \in \pi_1(B_{1,\text{sub}}^{A,4M\kappa}(x_k,k))} E_y\left(\exp\left\{ \int_0^A \overline{\xi}(X^\kappa(s), A(n - k) - s)\, ds \right\} \mathbb{1}\big\{ N(X^\kappa, A) > M\kappa A \big\} \right) \Bigg].
\end{aligned}
\tag{7.5}
$$

Now, by the Poisson tail estimate mentioned above and the fact that $\overline{\xi} < 2\delta A^d$, the second factor under the sum in (7.5) may be bounded from above by

$$
\left(e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\big\{ -M\kappa A[\log(M/2d) - 1] \big\} \right)^{n - |J|}.
\tag{7.6}
$$

Since, by Proposition 7.4 and our choice of κ (see the observation made prior to (7.3)), all blocks $B_1^{A,4M\kappa}(x,k)$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}_0$, $k \leq n$, are ε-sufficient at level n, we may conclude that all $y \in \pi_1(B_{1,\text{sub}}^{A,4M\kappa}(x_k,k))$ with $k \in J$ are in an ε-sufficient subpedestal at level n. Hence, using the binomial formula, we may estimate (7.5) from above by

$$\sum_{J \subset \{0,1,\dots,n-1\}} e^{A\varepsilon|J|} \left(e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\left\{ -M\kappa A[\log(M/2d) - 1] \right\} \right)^{n-|J|}$$

$$= \left(e^{A\varepsilon} + e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\left\{ -M\kappa A[\log(M/2d) - 1] \right\} \right)^n. \tag{7.7}$$

2. Summing over all possible sequences $(x_i)_{i \in \{1,2,\dots,n-1\}}$ compatible with a path Φ such that $\Phi(0) = 0$ and $N(\Phi, An) \leq M\kappa An$, and using (7.4–7.7) and Proposition 7.3, we obtain

$$E_0\left(\exp\left\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \right\} \mathbb{1}\left\{ N(X^\kappa, An) \leq M\kappa An \right\} \right)$$

$$\leq \sum_{x_1,x_2,\dots,x_{n-1}} E_0\left(\exp\left\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \right\} \mathbb{1}\left\{ N(X^\kappa, An) \leq M\kappa An \right\} \right.$$

$$\left. \times \prod_{k=1}^{n-1} \mathbb{1}\left\{ X^\kappa(kA) \in B_{1,\text{sub}}^{A,4M\kappa}(x_k, k) \right\} \right) \tag{7.8}$$

$$\leq \sum_{x_1,x_2,\dots,x_{n-1}} \prod_{k=0}^{n-1} \sup_{y \in B_{1,\text{sub}}^{A,4M\kappa}(x_k,k)} E_y\left(\exp\left\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \right\} \right)$$

$$\leq e^{C_3 n} \left(e^{A\varepsilon} + e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\left\{ -M\kappa A[\log(M/2d) - 1] \right\} \right)^n.$$

Combining (7.3–7.8), we get

$$\limsup_{\kappa \to \infty} \limsup_{n \to \infty} \frac{1}{An} \log E_0\left(\exp\left\{ \int_0^t \overline{\xi}(X^\kappa(s), t - s)\, ds \right\} \right)$$

$$\leq \frac{C_3}{A} + \frac{1}{A} \limsup_{\kappa \to \infty} \log\left(e^{A\varepsilon} + e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\left\{ -M\kappa A[\log(M/2d) - 1] \right\} \right) = \frac{C_3}{A} + \varepsilon. \tag{7.9}$$

Since $\varepsilon = 4\delta$, this yields the claim. ∎

7.3 Proof of Proposition 7.3

Proof. Write $\|\cdot\|$ for the ℓ^1-norm on $\mathbb{Z}^d$. Let $B_{1,\text{sub}}^{A,4M\kappa}(0,0), B_{1,\text{sub}}^{A,4M\kappa}(x_1,1), \dots, B_{1,\text{sub}}^{A,4M\kappa}(x_{n-1}, n-1)$ be a sequence of subpedestals that may be crossed by a path Φ with at most $M\kappa An$ jumps. Since Φ needs at least $(\|x_k - x_{k-1}\| - d)_+ 4M\kappa A$ jumps to go from $B_{1,\text{sub}}^{A,4M\kappa}(x_{k-1}, k-1)$ to $B_{1,\text{sub}}^{A,4M\kappa}(x_k, k)$, $k \in \{1, 2, \dots, n-1\}$, we obtain the bound

$$\sum_{k=1}^{n-1} (\|x_k - x_{k-1}\| - d)_+ \leq \frac{M\kappa An}{4M\kappa A} = \frac{n}{4}, \tag{7.10}$$

which implies that

$$\sum_{k=1}^{n-1} \|x_k - x_{k-1}\| \leq \frac{(1 + 4d)n}{4}. \tag{7.11}$$

As shown in Hardy and Ramanujan [7] and Erdös [2], there are $a, b > 0$ such that the number of integer-valued sequences $(a_k)_{k \in \mathbb{N}}$ such that $\sum_{k \in \mathbb{N}} a_k \leq (1 + 4d)n/4$ is bounded from above by $ane^{b\sqrt{n}}$. To conclude, define $a_k = \|x_k - x_{k-1}\|$, $k \in \{1, 2, \ldots, n-1\}$, and note that the sequence $(a_k)_{k \in \{1,2,\ldots,n-1\}}$ determines the sequence $(x_k)_{k \in \{0,1,\ldots,n-1\}}$ uniquely when it is known for all $k \in \{1, 2, \ldots, n-1\}$ and all $j \in \{1, 2, \ldots, d\}$ whether $x_k(j) - x_{k-1}(j)$ is positive, zero or negative. Consequently, the number of different subpedestals $B_{1,\text{sub}}^{A,4M\kappa}(0, 0), B_{1,\text{sub}}^{A,4M\kappa}(x_1, 1), \ldots, B_{1,\text{sub}}^{A,4M\kappa}(x_{n-1}, n-1)$ that may be crossed by a path Φ with at most $M\kappa An$ jumps is bounded from above by $3^{dn}ane^{b\sqrt{n}} \leq e^{C_3 n}$ for some $C_3 > 0$. ∎

7.4 Proof of Proposition 7.4

The proof of Proposition 7.4 is given in Section 7.5 subject to Lemmas 7.5–7.6 below, which are stated in Sections 7.4.1–7.4.2. The proof of the first lemma is given in Section 7.4.1, the proof of the second lemma is deferred to Appendix B.

7.4.1 A time-dependent Feynman-Kac estimate

Recall (2.4). Abbreviate

$$Q^{\kappa \log \kappa} = (-\kappa \log \kappa, \kappa \log \kappa)^d \cap \mathbb{Z}^d. \tag{7.12}$$

Lemma 7.5. *Fix $A > 1$ and $m > 0$ such that $Am \in \mathbb{N}$. There is a $\kappa_0 = \kappa_0(M, A)$ such that, ξ-a.s. for all $x \in \mathbb{Z}^d$,*

$$\log E_x\left(\exp\left\{\int_0^A \overline{\xi}(X^\kappa(s), A - s)\, ds\right\} \mathbb{1}\{N(X^\kappa, A) \leq M\kappa A\}\right) \leq \frac{\kappa}{m} \sum_{k=1}^{Am} \lambda_1(\overline{\xi}_k/\kappa), \quad \kappa \geq \kappa_0, \tag{7.13}$$

where $\lambda_1(\overline{\xi}_k/\kappa)$ is the top of the spectrum of the operator $\Delta + \frac{1}{\kappa} \sup_{r \in [(k-1)/m, k/m)} \overline{\xi}(\cdot, r)$, $k \in \{1, 2, \ldots, Am\}$.

Proof. We give the proof for $x = 0$. The proof for $x \in \mathbb{Z}^d \setminus \{0\}$ goes along the same lines. First note that we may rewrite the expectation in the left-hand side of (7.13) as

$$E_0\left(\exp\left\{\frac{1}{\kappa} \int_0^{\kappa A} \overline{\xi}(X(s), A - s/\kappa)\, ds\right\} \mathbb{1}\{N(X, \kappa A) \leq M\kappa A\}\right), \tag{7.14}$$

where X is simple random walk with step rate $2d$. Furthermore, there is a $\kappa_0 = \kappa_0(M, A)$ such that $M\kappa A \leq \kappa \log \kappa$ for all $\kappa \geq \kappa_0$. Hence, by the Markov property of X applied at times $k\kappa/m$, $k \in \{1, 2, \ldots, Am\}$, we may estimate (7.14) from above by

$$E_0\left(\exp\left\{\frac{1}{\kappa} \int_0^{\kappa A} \overline{\xi}(X(s), A - s/\kappa)\, ds\right\} \mathbb{1}\{X([0, \kappa A]) \subseteq Q^{\kappa \log \kappa}\}\right)$$

$$\leq \prod_{k=1}^{Am} \sup_{\substack{x \in \mathbb{Z}^d \\ \|x\|_\infty < \kappa \log \kappa}} E_x\left(\exp\left\{\frac{1}{\kappa} \int_0^{\kappa/m} \overline{\xi}(X(s), k/m - s/\kappa)\, ds\right\} \mathbb{1}\{X([0, \kappa/m]) \subseteq Q^{\kappa \log \kappa}\}\right). \tag{7.15}$$

Next, for $k \in \{1, 2, \ldots, Am\}$ define

$$\overline{\xi}_k(x) = \sup_{r \in [(k-1)/m, k/m)} \overline{\xi}(x, r), \qquad x \in \mathbb{Z}^d. \tag{7.16}$$

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Then (7.15) is at most

$$\prod_{k=1}^{Am} \sup_{x\in\mathbb{Z}^d//\|x\|_\infty<\kappa\log\kappa} E_x\left(\exp\left\{\frac{1}{\kappa}\int_0^{\kappa/m}\overline{\xi}_k(X(s))\,ds\right\}\mathbb{1}\{X([0,\kappa/m])\subseteq Q^{\kappa\log\kappa}\}\right). \quad (7.17)$$

For each $k\in\{1,2,\dots,Am\}$, each expectation under the product in (7.17) is a solution of the equation

$$\begin{cases} \frac{\partial u_k}{\partial t}(x,t)=\left[(\Delta+\frac{1}{\kappa}\overline{\xi}_k(x))u_k\right](x,t), \\ u_k(x,0)=1, \end{cases} \qquad \|x\|_\infty<\kappa\log\kappa, t\geq 0, \qquad (7.18)$$

with Dirichlet boundary conditions evaluated at time κ/m. However, on any finite subset of $\mathbb{Z}^d$ the operator $\Delta+\frac{1}{\kappa}\overline{\xi}_k$ is a self-adjoint matrix. Therefore, by the spectral representation theorem, we may rewrite each expectation under the product in (7.17) as

$$\sum_{j=1}^{|Q^{\kappa\log\kappa}|} e^{(\kappa/m)\lambda_j^D(\overline{\xi}_k/\kappa)}\langle v_j^k,\mathbb{1}_{Q^{\kappa\log\kappa}}\rangle\, v_j^k(x), \quad (7.19)$$

where $\lambda_j^D(\overline{\xi}_k/\kappa)$ is the j-th largest eigenvalue of $\Delta+\overline{\xi}_k/\kappa$ with Dirichlet boundary conditions on $Q^{\kappa\log\kappa}$, $j\in\{1,2,\dots,|Q^{\kappa\log\kappa}|\}$, and the v_j^k, $j\in\{1,2,\dots,|Q^{\kappa\log\kappa}|\}$, form an orthonormal system of eigenvectors such that, for all $k\in\{1,2,\dots,Am\}$,

$$\mathbb{R}^{|Q^{\kappa\log\kappa}|}=\ker\left(e^{\Delta+\overline{\xi}_k/\kappa}\right)\oplus\text{span}\{v_j^k,j\in\{1,2,\dots,|Q^{\kappa\log\kappa}|\}\}. \quad (7.20)$$

(Since $e^{\Delta+\overline{\xi}_k/\kappa}$ is a strictly positive operator, $\ker(e^{\Delta+\overline{\xi}_k/\kappa})=\{0\}$.) In particular, for each $k\in\{1,2,\dots,Am\}$ there is a sequence of real-valued numbers $(\mu_j^k)_{j\in\{1,2,\dots,|Q^{\kappa\log\kappa}|\}}$ such that $\sum_{j=1}^k\mu_j^k v_j^k=\mathbb{1}_{Q^{\kappa\log\kappa}}$. Inserting the above representation of $\mathbb{1}_{Q^{\kappa\log\kappa}}$ into (7.19), we see that (7.19) is bounded from above by

$$e^{(\kappa/m)\lambda_1^D(\overline{\xi}_k/\kappa)}. \quad (7.21)$$

Combining (7.14)–(7.21), we get

$$\log E_x\left(\exp\left\{\int_0^A\overline{\xi}(X^\kappa(s),A-s)\,ds\right\}\mathbb{1}\{N(X^\kappa;A)\leq M\kappa A\}\right)\leq\frac{\kappa}{m}\sum_{k=1}^{Am}\lambda_1^D(\overline{\xi}_k/\kappa). \quad (7.22)$$

Finally, by the Rayleigh-Ritz principle we have that $\lambda_1^D(\overline{\xi}_k/\kappa)\leq\lambda_1(\overline{\xi}_k/\kappa)$, where $\lambda_1(\overline{\xi}_k/\kappa)$ is the top of the spectrum of $\Delta+\overline{\xi}_k/\kappa$. ∎

7.4.2 A spectral estimate

Let $(B(x))_{x\in\mathbb{Z}^d}$ be an arbitrary partition of $\mathbb{Z}^d$ into finite boxes. Let $\langle\cdot,\cdot\rangle$ be the scalar product on $\mathbb{R}^B$ and on $\ell^2(\mathbb{Z}^d)$. Let $V\colon\mathbb{Z}^d\to\mathbb{R}$ be bounded such that there is a $\delta>0$ for which

$$\frac{1}{|B(x)|}\sum_{y\in B(x)}V(y)\leq 2\delta\qquad x\in\mathbb{Z}^d. \quad (7.23)$$

The proof of the following lemma is deferred to Appendix B.

Lemma 7.6. *Subject to* (7.23), *there is a $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$,*

$$\sup_{\substack{f \in l^2(\mathbb{Z}^d) \\ \|f\|_2 = 1}} \left\langle \left(\Delta + \frac{1}{\kappa} V \right) f, f \right\rangle \leq 4 \frac{1}{\kappa} \delta. \tag{7.24}$$

Lemma 7.6 and the Rayleigh-Ritz principle yield that the top of the spectrum of $\Delta + \frac{1}{\kappa} V$ is bounded from above by $4 \frac{1}{\kappa} \delta$ for $\kappa \geq \kappa_0$.

7.5 Completion of the proof of Proposition 7.4

Proof. Fix $\delta > 0$, $A > 1$ and $m > 1$. By Lemma 7.5, there is a $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$ and ξ-a.s. for all $x \in \mathbb{Z}^d$,

$$
\begin{aligned}
&\log E_x \left(\exp \left\{ \int_0^A \overline{\xi}(X^\kappa(s), A - s)\, ds \right\} \mathbb{1}\{ N(X^\kappa; A) \leq M\kappa A \} \right) \\
&\leq \frac{\kappa}{m} \sum_{k=1}^{Am} \lambda_1(\overline{\xi}_k / \kappa), \qquad \kappa \geq \kappa_0 = \kappa_0(M, A).
\end{aligned}
\tag{7.25}
$$

Next, by Lemma 7.6 with $V = \overline{\xi}_k$, $k \in \{1, 2, \ldots, Am\}$ (recall (7.16)) and $B(x) = \pi_1(B_1^A(x, 0))$ (recall (1.13); π_1 denotes the projection onto the spatial coordinates), there is a $\kappa_1 > 0$ such that, for all $\kappa \geq \kappa_1$ and all $k \in \{1, 2, \ldots, Am\}$,

$$\lambda_1(\overline{\xi}_k / \kappa) \leq 4 \frac{1}{\kappa} \delta. \tag{7.26}$$

This shows that, ξ-a.s. for $\kappa \geq \max\{\kappa_0, \kappa_1\}$, any block $B_1^{A, 4M\kappa}(x, 0)$, $x \in \mathbb{Z}^d$, is ε-sufficient at level 1. The stationarity of ξ in time completes the proof. ∎

A   Proof of Lemmas 4.3–4.4

In this section we prove two lemmas that were used in Section 4.

A.1   Proof of Lemma 4.3

Proof. Our first observation is that

$$\limsup_{t \to \infty} \frac{1}{t} \log E_0 \left(\exp \left\{ \sum_{i=1}^n \beta_i \sum_{x \in I_i} l_t(X^\kappa, x) \right\} \right) \leq \mu, \tag{A.1}$$

where

$$\mu = \sup_{\substack{f \in l^2(\mathbb{Z}): \\ \|f\|_2 = 1,\, f \geq 0}} \mu(f) \tag{A.2}$$

with

$$\mu(f) = \left(f, \left[\kappa \Delta + \sum_{i=1}^n \beta_i \mathbb{1}_{I_i} \right] f \right) = \sum_{i=1}^n \beta_i \sum_{x \in I_i} f^2(x) - \kappa \sum_{x \in \mathbb{Z}} [f(x+1) - f(x)]^2. \tag{A.3}$$

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Indeed, this follows from the large deviation principle for the occupation time measure of one-dimensional simple random walk on $\mathbb{Z}$ with jump rate 2κ (which is the continuous-time Markov process with generator $\kappa\Delta$) in combination with Varadhan's lemma (see [9, Chapters 3–4]). A formal proof proceeds by truncating $\mathbb{Z}$ to a large finite torus, wrapping the random walk around the torus, deriving the claim for a fixed torus size, letting the torus size tend to infinity, and showing that the variational formula on the finite torus converges to the variational formula on $\mathbb{Z}$. The details are standard and are left to the reader (see [9, Chapter 8]).

We claim that μ is the largest eigenvalue of $\kappa\Delta + \sum_{i=1}^n \beta_i \mathbb{1}_{I_i}$. Indeed, by [8, Theorem 2.2] the operator $\kappa\Delta + \sum_{i=1}^n \beta_i \mathbb{1}_{I_i}$ has at least one eigenvalue. Consequently, the min-max principle [15, Theorem XIII.1] yields the claim. The inequality in (4.21) now follows from [17, Corollary 1.4]. ∎

A.2 Proof of Lemma 4.4

Proof. Let $t, \kappa > 0$ and let X^κ be one-dimensional random walk with step rate $2\kappa > 0$. We first show that for all $C > 0$,

$$P_0\left(\sup_{0 \le s \le t} |X^\kappa(s)| \ge C\sqrt{\kappa t}\right) \le 2e^{-\frac{C^2\sqrt{\kappa t}}{2(C+3\sqrt{\kappa t})}}. \qquad (A.4)$$

The proof is based on a discretization argument in combination with Bernstein's inequality. Fix $n \in \mathbb{N}$ with $n \gg \kappa$, and define

$$q^n(y) = \begin{cases} 1 - \frac{2\kappa}{n}, & y = 0 \\ \frac{\kappa}{n}, & y = \pm 1 \\ 0, & \text{elsewhere.} \end{cases} \qquad (A.5)$$

Let $X^{(n)} = (X^{(n)}(t))_{t \ge 0}$ be the discrete-time random walk with jump distribution q^n and jump times k/n, $k \in \mathbb{N}$. Then, for each $t > 0$, $(X^{(n)}(s))_{0 \le s \le t}$ converges weakly as $n \to \infty$ to $(X^\kappa(s))_{0 \le s \le t}$ in the Skorokhod space $D([0,t], \mathbb{Z})$. Since $X^{(n)}$ is unlikely to move in a short time interval, uniformly in n, it is enough to prove (A.4) for $X^{(n)}$ with n fixed. To that end, let $k \in \mathbb{N}$ be such that $k/n \le t < (k+1)/n$. Note that, because $X^{(n)}$ is a martingale, Doob's maximal inequality and Bernstein's inequality yield

$$P_0\left(\sup_{0 \le s \le t} X^{(n)}(s) \ge C\sqrt{\kappa t}\right) \le \exp\left\{-\frac{C^2\kappa t}{2(C\sqrt{\kappa t} + 3\kappa t)}\right\} \qquad \forall\, C > 0. \qquad (A.6)$$

The same inequality is valid for the probability of $\sup_{0 \le s \le t}[-X^{(n)}(s)] \ge C\sqrt{\kappa t}$, which yields the claim in (A.4).

Next, note that if X^κ leaves the spatial part of a space-time block $B_1(x, k; \kappa)$, then there is at least one coordinate $j \in \{1, \dots, d\}$ such that $\pi_j(X^\kappa(s)) \notin [\sqrt{\kappa}x(j)A, \sqrt{\kappa}(x(j)+1)A)$ for some $s \in [kA, (k+1)A)$, where $\pi_j(X^\kappa)$ denotes the projection of X^κ onto the j-th coordinate. In particular, if X^κ visits l_i^κ $(\kappa, 1)$-blocks with $l_i^\kappa > d$ in the time interval $[(i-1)A, iA)$, then there is at least one coordinate that visits at least l_i^κ/d one-dimensional $(\kappa, 1)$-blocks, i.e., blocks of the form $[\sqrt{\kappa}xA, \sqrt{\kappa}(x+1)A) \times [kA, (k+1)A)$, $x \in \mathbb{Z}$. Consequently, without loss of generality we may assume that X^κ is one-dimensional simple random walk with step rate 2κ.

Given $l_1^\kappa, \ldots, l_{t/A}^\kappa \in \mathbb{N}$, we say that X^κ has label $(l_1^\kappa, \ldots, l_{t/A}^\kappa)$ when X^κ crosses l_i^κ $(\kappa, 1)$-blocks in the time interval $[(i-1)A, iA)$, $1 \leq i \leq t/A$.

Next, fix $C_7 > 0$ and let $k_*(\kappa) \geq C_7 t$, and note that

$$P_0\Big(X^\kappa \text{ crosses } k_*(\kappa)\ (\kappa, 1)\text{-blocks}\Big) = \sum_{\substack{(l_i^\kappa)_{i=1}^{t/A}: \\ \sum l_i^\kappa = k_*(\kappa)}} P_0\Big(X^\kappa \text{ has label } \big(l_1^\kappa, \ldots, l_{t/A}^\kappa\big)\Big). \quad \text{(A.7)}$$

Using the Markov property and the fact that a path crossing l_i^κ $(\kappa, 1)$-blocks has to travel a distance at least $(l_i^\kappa - 2)\sqrt{\kappa}A/2$, we may further estimate each summand in the right-hand side of (A.7) by

$$E_0\Big(\mathbb{1}\{X^\kappa \text{ has label } (l_1^\kappa, \ldots, l_{t/A-1}^\kappa)\}P_{X^\kappa(t-A)}\big(X^\kappa \text{ has label } l_{t/A}^\kappa\big)\Big)$$

$$\leq E_0\Big(\mathbb{1}\{X^\kappa \text{ has label } (l_1^\kappa, \ldots, l_{t/A-1}^\kappa)\}P_{X^\kappa(t-A)}\Big(\sup_{0 \leq s \leq A} |X^\kappa(s)| \geq (l_i^\kappa - 2)\sqrt{\kappa}A/2\Big)\Big). \quad \text{(A.8)}$$

To proceed, note that, by (A.4),

$$P_{X^\kappa(t-A)}\Big(\sup_{0 \leq s \leq A} |X^\kappa(s)| \geq (l_i^\kappa - 2)\sqrt{\kappa}A/2\Big) \leq \begin{cases} 2\exp\Big\{-\frac{1}{4}\frac{(l_i^\kappa-2)^2 A\sqrt{\kappa A}}{(l_i^\kappa-2)\sqrt{A}+6\sqrt{\kappa A}}\Big\}, & l_i^\kappa \geq 3 \\ 1, & l_i^\kappa \leq 2. \end{cases} \quad \text{(A.9)}$$

An iteration of the estimates in (A.8) yields that the left-hand side of (A.7) is bounded from above by

$$\sum_{\substack{(l_i^\kappa)_{i=1}^{t/A}: \\ \sum l_i^\kappa = k_*(\kappa)}} \prod_{\substack{i=1 \\ l_i^\kappa \geq 3}}^{t/A} 2\exp\Big\{-\frac{1}{4}\frac{(l_i^\kappa - 2)^2 A\sqrt{\kappa A}}{(l_i^\kappa - 2)\sqrt{A} + 6\sqrt{\kappa A}}\Big\}. \quad \text{(A.10)}$$

We have

$$\sum_{\substack{i=1: \\ l_i^\kappa \geq 3}}^{t/A} \frac{(l_i^\kappa - 2)^2}{(l_i^\kappa - 2)\sqrt{A} + 6\sqrt{\kappa A}} \geq \sum_{\substack{i=1: \\ l_i^\kappa \geq 3}}^{t/A} \frac{(l_i^\kappa - 2)}{\sqrt{A} + 6\sqrt{\kappa A}}. \quad \text{(A.11)}$$

Moreover,

$$\sum_{\substack{i=1: \\ l_i^\kappa \geq 3}}^{t/A} (l_i^\kappa - 2) \geq k_*(\kappa) - \frac{4t}{A}. \quad \text{(A.12)}$$

Inserting (A.11–A.12) into (A.10), noting that [7, 2]

$$\exists\, a, b > 0: \quad \text{number of summands in right-hand side of (A.7)} \ \leq ae^{b\sqrt{k_*(\kappa)}}/k_*(\kappa), \quad \text{(A.13)}$$

and choosing C_7 and κ large enough ($C_7 > 4$ is sufficient), we get that the right-hand side of (A.7) is at most $e^{-C_8 A k_*(\kappa)}$. Here, C_8 is such that for all $k_*(\kappa) \geq C_7/A$ the inequality $(k_*(\kappa) - 4t/A)/(1/\sqrt{\kappa} + 6) \geq C_8 k_*(\kappa)$ holds, which for $\kappa \geq 1$ is fulfilled when $C_7(1 - 7C_8) \geq 4$. This yields the claim in (4.22). ∎

B   Proof of Lemma 7.6

In Sections B.1–B.3 we prove a lemma that was used in Section 7. The proof is inspired by [1, Theorem 12].

B.1   Neumann boundary conditions

In this section we recall the definition and some properties of the discrete Laplacian with Neumann boundary conditions. For further details we refer the reader to [11].

Fix $x \in \mathbb{Z}^d$, $A > 1$ and define the matrix $M_{B(x)}$ as

$$M_{B(x)}(y,z) = \begin{cases} 1, & \text{if } y, z \in B(x), \|y - z\| = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{B.1}$$

and the number of neighbors of y in $B(x)$ as

$$n_{B(x)}(y) = |\{z \in B(x) \colon \|y - z\| = 1\}|. \tag{B.2}$$

Definition B.1. *The Neumann Laplacian $\Delta_{B(x)}$ on $B(x)$ is defined via the formula*

$$\Delta_{B(x)} = M_{B(x)} - n_{B(x)}, \tag{B.3}$$

where $n_{B(x)}$ is the multiplication operator with the function $n_{B(x)}$.

Remark B.2. *The quadratic form associated with $\Delta_{B(x)}$ is given by*

$$\left\langle \Delta_{B(x)} f, g \right\rangle = -\frac{1}{2} \sum_{\substack{y,z \in B(x) \\ \|y-z\|=1}} [f(y) - f(z)]\,[g(y) - g(z)], \qquad f, g \in \mathbb{R}^{B(x)}. \tag{B.4}$$

($\Delta_{B(x)}$ does not see that $B(x)$ is imbedded in $\mathbb{Z}^d$, which is why it is sometimes referred to as the graph Laplacian on $B(x)$.)

Lemma B.3. *The following properties hold for all $x \in \mathbb{Z}^d$ and $A > 1$.*
(a) $\langle \Delta_{B(x)} f, f \rangle \leq 0$ for all $f \in \mathbb{R}^{B(x)}$.
(b) $\Delta_{B(x)}$ is self-adjoint.
(c) $\ker(\Delta_{B(x)}) = \mathbb{R}\mathbb{1}$, where $\mathbb{1}$ is the vector in $\mathbb{R}^{B(x)}$ with all entries equal to one.
(d) For all $f \in \ell^2(\mathbb{Z}^d)$,
$$\langle \Delta f, f \rangle \leq \sum_{x \in \mathbb{Z}^d} \left\langle \Delta_{B(x)} f^{B(x)}, f^{B(x)} \right\rangle, \tag{B.5}$$

where $f^{B(x)}$ is the restriction of f to $B(x)$.

Proof. Fix $x \in \mathbb{Z}^d$ and $A > 1$.
(a) and (b) are consequences of Remark B.2.
(c) From Remark B.2 it is clear that constant functions are in the kernel of $\Delta_{B(x)}$. For the reverse direction, let $f \in \ker(\Delta_{B(x)})$. Again by Remark B.2,

$$0 = \left\langle \Delta_{B(x)} f, f \right\rangle = -\frac{1}{2} \sum_{\substack{y,z \in B(x) \\ \|y-z\|=1}} [f(y) - f(z)]^2. \tag{B.6}$$

Hence, for all $y \in B(x)$ we have that $f(y) = f(z)$ for all z such that $\|y - z\| = 1$, $z \in B(x)$.

(d) Let $f \in \ell^2(\mathbb{Z}^d)$. Then

$$
\begin{aligned}
-2\langle \Delta f, f \rangle &= \sum_{\substack{y,z\in\mathbb{Z}^d \\ \|y-z\|=1}} [f(y) - f(z)]^2 = \sum_{x\in\mathbb{Z}^d} \sum_{y\in B(x)} \sum_{\substack{z\in\mathbb{Z}^d \\ \|y-z\|=1}} [f(y) - f(z)]^2 \\
&\geq \sum_{x\in\mathbb{Z}^d} \sum_{y\in B(x)} \sum_{\substack{z\in B(x) \\ \|y-z\|=1}} [f(y) - f(z)]^2 = -2 \sum_{x\in\mathbb{Z}^d} \left\langle \Delta_{B(x)} f^{B(x)}, f^{B(x)} \right\rangle,
\end{aligned}
\tag{B.7}
$$

where the second equality uses that $(B(x))_{x\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$, while the third equality follows from Remark B.2. ∎

B.2 Proof of Lemma 7.6 subject to a further lemma

Let $\|\cdot\|_2$ stand for both the Euclidean norm on $\mathbb{R}^B$ and the ℓ^2-norm on $\ell^2(\mathbb{Z}^d)$.

Lemma B.4. *Subject to* (7.23), *there is a $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$, all $f \in \mathbb{R}^{B(x)}$ and all $x \in \mathbb{Z}^d$,*

$$
\left\langle (\Delta_{B(x)} + \frac{1}{\kappa}V)f, f \right\rangle \leq 4\frac{1}{\kappa}\delta\|f\|_2^2.
\tag{B.8}
$$

The proof of Lemma B.4 is deferred to Section B.3. First we complete the proof of Lemma 7.6 subject to Lemma B.4.

Proof. Let $f \in \ell^2(\mathbb{Z}^d)$ and $\kappa \geq \kappa_0$, where κ_0 is chosen according to Lemma B.4. Then, by Lemma B.3(d) and the fact that $(B(x))_{x\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$, we may estimate

$$
\left\langle (\Delta + \frac{1}{\kappa}V)f, f \right\rangle \leq \sum_{x\in\mathbb{Z}^d} \left\langle \left(\Delta_{B(x)} + \frac{1}{\kappa}V\right) f^{B(x)}, f^{B(x)} \right\rangle.
\tag{B.9}
$$

Since $f = \sum_{x\in\mathbb{Z}^d} f^{B(x)}$, we have $\|f\|^2 = \sum_{x\in\mathbb{Z}^d} \|f^{B(x)}\|_2^2$. Combining (B.8–B.9), we get the claim. ∎

B.3 Proof of Lemma B.4

Proof. Fix $x \in \mathbb{Z}^d$ and $A > 1$. First recall that, by Lemma B.3(c), $\ker(\Delta_{B(x)}) = \mathbb{R}\mathbb{1}$, so that we can write each $f \in \mathbb{R}^{B(x)}$ as $f = \alpha\mathbb{1} + g$, $\alpha \in \mathbb{R}$, $g \in (\mathbb{R}\mathbb{1})^\perp$. Therefore, using that $\Delta_{B(x)}$ is self-adjoint, see Lemma B.3(b) and hence symmetric, we obtain that

$$
\left\langle \left(\Delta_{B(x)} + \frac{1}{\kappa}V\right)f, f \right\rangle = \left\langle \Delta_{B(x)}g, g \right\rangle + \frac{1}{\kappa}\left[\alpha^2\langle V\mathbb{1}, \mathbb{1}\rangle + 2\alpha\langle V\mathbb{1}, g\rangle + \langle Vg, g\rangle\right].
\tag{B.10}
$$

Using that the unit sphere intersected with $(\mathbb{R}\mathbb{1})^\perp$ is compact, that $\Delta_{B(x)}$ is negative on $(\mathbb{R}\mathbb{1})^\perp$, see Lemma B.3(a), and that the scalar product is continuous, we deduce that there is an $\eta > 0$ such that $\langle \Delta_{B(x)}h, h \rangle \leq -\eta$ for all $h \in (\mathbb{R}\mathbb{1})^\perp$ with $\|h\|_2 = 1$. Hence we may estimate the right-hand side of (B.10) from above by

$$
-\eta\|g\|_2^2 + \frac{1}{\kappa}\left[\alpha^2\langle V\mathbb{1}, \mathbb{1}\rangle + 2\alpha\langle V\mathbb{1}, g\rangle + \langle Vg, g\rangle\right].
\tag{B.11}
$$

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Next, by (7.23), we have $\langle V \mathbb{1}, \mathbb{1} \rangle = \sum_{y \in B(x)} V(y) \leq 2\delta |B(x)|$. An additional application of the Cauchy-Schwarz inequality shows that (B.11) is at most

$$-\eta \|g\|_2^2 + \frac{1}{\kappa}\big[\alpha^2 2\delta |B(x)| + 2\alpha\|V\mathbb{1}\|_2\|g\|_2 + \|Vg\|_2\|g\|_2\big]. \tag{B.12}$$

Using the bound $\|Vg\|_2 \leq \|V\|_\infty \|g\|_2$ (which also holds for g replaced by $\mathbb{1}$) and the assumption that V is bounded, we may further estimate (B.12) from above by

$$-\eta \|g\|_2^2 + \frac{1}{\kappa}\big[\alpha^2 2\delta |B(x)| + 2\alpha\|V\|_\infty\|\mathbb{1}\|_2\|g\|_2 + \|V\|_\infty\|g\|_2^2\big]. \tag{B.13}$$

For any $a, b \in \mathbb{R}$ and $\gamma > 0$ we have the inequality $2ab \leq \gamma a^2 + b^2/\gamma$. Pick $a = \alpha\|V\|_\infty\|\mathbb{1}\|_2$ and $b = \|g\|_2$. Then we may further estimate (B.13) from above by

$$\begin{aligned}
&-\eta \|g\|_2^2 + \frac{1}{\kappa}\big[\alpha^2 2\delta |B(x)| + \gamma\alpha^2\|V\|_\infty^2\|\mathbb{1}\|_2^2 + \|g\|_2^2/\gamma + \|g\|_2^2\big] \\
&= \|g\|_2^2\Big[-\eta + \frac{1}{\kappa}\Big(1 + \frac{1}{\gamma}\Big)\Big] + \frac{1}{\kappa}\big[2\delta + \gamma\|V\|_\infty^2\big]\alpha^2\|\mathbb{1}\|_2^2,
\end{aligned} \tag{B.14}$$

where we use that $\|\mathbb{1}\|_2^2 = |B(x)|$. Now pick $\gamma = 2\delta/\|V\|_\infty^2$ and note that, for κ large enough so that $\frac{1}{\kappa}(1 + 1/\gamma - 4\delta) \leq \eta$, we have $-\eta + \frac{1}{\kappa}(1 + 1/\gamma) \leq 4\frac{1}{\kappa}\delta$. Therefore, for κ large enough we may estimate the right-most term in (B.14) from above by

$$4\frac{1}{\kappa}\delta\Big[\|g\|_2^2 + \alpha^2\|\mathbb{1}\|_2^2\Big] = 4\frac{1}{\kappa}\delta\|f\|_2^2. \tag{B.15}$$

∎

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