The parabolic Anderson model in a dynamic random environment: space-time ergodicity for the quenched Lyapunov exponent

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Abstract

We continue our study of the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, and ξ plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$.

We assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$. In earlier work [6], [3] we established a number of basic properties of $\kappa \mapsto \lambda_0(\kappa)$ under certain mild space-time mixing and noisiness assumptions on ξ. In particular, we showed that the limit exists ξ-a.s., is finite and continuous on $[0,\infty)$, is globally Lipschitz on $(0,\infty)$, is not Lipschitz at 0, and satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$.

In the present paper we show that $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \mathbb{E}(\xi(0,0))$ under an additional space-time mixing condition on ξ we call Gärtner-hyper-mixing. This result, which completes our study of the quenched Lyapunov exponent for general ξ, shows that the parabolic Anderson model exhibits space-time ergodicity in the limit of large diffusivity. This fact is interesting because there are choices of ξ that are Gärtner-hyper-mixing for which the *annealed Lyapunov exponent* $\lambda_1(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(u(0,t))$ is infinite on $[0,\infty)$, a situation that is referred to as *strongly catalytic behavior*. Our proof is based on a *multiscale analysis* of ξ, in combination with discrete rearrangement inequalities for local times of simple random walk and spectral bounds for discrete Schrödinger operators.

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