The parabolic Anderson model in a dynamic random environment: basic properties of the quenched Lyapunov exponent

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Abstract

In this paper we study the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where the u-field and the ξ-field are $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, and Δ is the discrete Laplacian. The ξ-field plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$. Our goal is to prove a number of *basic properties* of the solution u under assumptions on ξ that are as weak as possible. These properties will serve as a jump board for later refinements.

Throughout the paper we assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Under a mild assumption on the tails of the distribution of ξ, we show that the solution to the parabolic Anderson equation exists and is unique for all $\kappa \in [0,\infty)$. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t}\log u(0,t)$. It was shown in Gärtner, den Hollander and Maillard [7] that this exponent exists and is constant ξ-a.s., satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$, and is such that $\kappa \mapsto \lambda_0(\kappa)$ is globally Lipschitz on $(0,\infty)$ outside any neighborhood of 0 where it is finite. Under certain weak space-time mixing assumptions on ξ, we show the following properties: (1) $\lambda_0(\kappa)$ does not depend on the initial condition u_0; (2) $\lambda_0(\kappa) < \infty$ for all $\kappa \in [0,\infty)$; (3) $\kappa \mapsto \lambda_0(\kappa)$ is continuous on $[0,\infty)$ but not Lipschitz at 0. We further conjecture: (4) $\lim_{\kappa\to\infty}[\lambda_p(\kappa) - \lambda_0(\kappa)] = 0$ for all $p \in \mathbb{N}$, where $\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt}\log \mathbb{E}([u(0,t)]^p)$ is the p-th *annealed Lyapunov exponent*. (In [7] properties (1), (2) and (4) were not addressed, while property (3) was shown under much more restrictive assumptions on ξ.) Finally, we prove that our weak space-time mixing conditions on ξ are satisfied for several classes of interacting particle systems.

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