Hydrodynamics of a non-gradient model for collective dynamics

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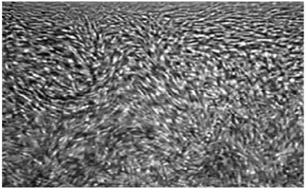
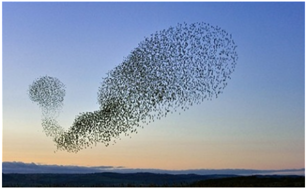
Plan of the talk

Plan of the talk

Collective motion & Active matter

Collective behavior can be observed among numerous animal species

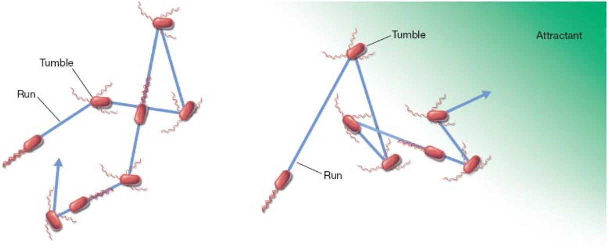





→ Classical representation : *Individual Based Models* (IBM) built around ***active matter***.

Active Matter

System composed of many individuals, maintained out of equilibrium by an *energy influx at the individual level*.



(a) No attractant present: Random movement (b) Attractant present: Directed movement

Two types of phenomena can arise in active matter models :

$\rightarrow$ *Alignment phase transition*

$\rightarrow$ *Motility Induced Phase Separation (MIPS)*.

Alignment phase transition

Alignment

Alignment dynamics will create *groups of particles moving together* and adapting their speed

Original model by Vicsek&al. (1995, *Novel type of phase transition in a system of self driven particles*)

$\mapsto N = \rho L^2$ particles move in the periodic domain $[0, L]^2$, with speed $v_i(t) = v \overrightarrow{e}_{\theta_i(t)}$

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t)\Delta t \\ \theta_i(t+1) = \langle \theta(t) \rangle_r + \xi_\eta \end{cases}$$

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Numerical results on related models

- Solon & Tailleur, 2013
- Solon & al., 2015

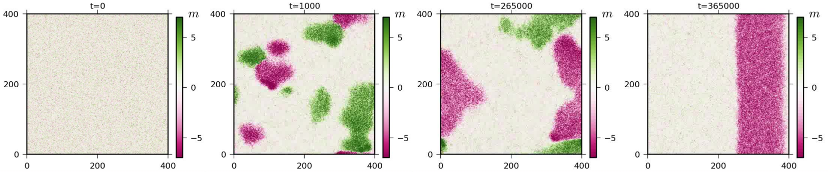


Figure: Emergence of global order for an alignment dynamics. From *Flocking with discrete symmetry, the 2d active Ising model*, Solon & Tailleur 2015.

Exact works

Several analysis and PDE based papers, based on *mean-field interactions* :

$\mapsto$ Each particle interacts with a large number of neighbors

Degond, Motsch, 2007

Bolley, Carrillo, Canizo, 2011

Motility induced phase separation

MIPS

Particles will tend to *accumulate where they move more slowly*

- MIPS can occur when the particle's velocity depends on the local density
- MIPS usually does not occur in models with alignment

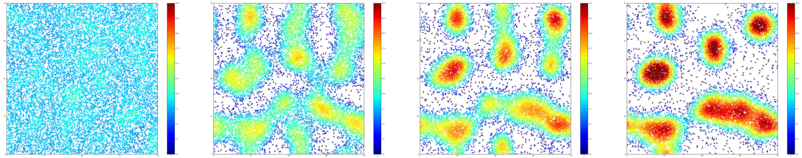


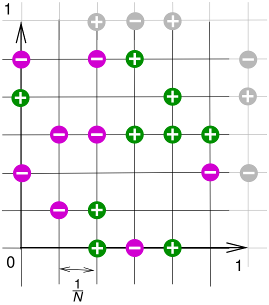
Figure: Coarsening effect in active matter. From Cates & Tailleur 2012.

Plan of the talk

Description of the particle system

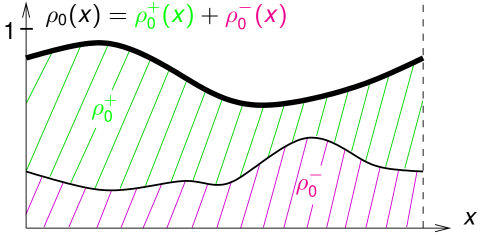
Two types of particles ($+$ and $-$) evolve on the **periodic square lattice** $\mathbb{T}_N^2 = \{0, \frac{1}{N}, ..., \frac{N-1}{N}\}^2$

- For each site $x \in \mathbb{T}_N^2$, we define $\eta_x \in \{-1, 0, 1\}$.
  $\mapsto \eta_x = 0$, empty site
  $\mapsto \eta_x = \pm 1$ site occupied by a particle $\pm$
- We let $\eta_x = \eta_x^+ - \eta_x^-$, where $\eta_x^\pm = 1$ iff x is occupied by a $\pm$ particle.



Initial configuration

- **Initial macroscopic smooth profiles** ρ_0^+ and ρ_0^-, $[0,1]^2 \to [0,1]$.
- $\rho_0 = \rho_0^+ + \rho_0^-$ is the initial particle density



- Initial ***local equilibrium*** : independently for any $x \in \mathbb{T}_N^2$

$$\eta_x(0) = \begin{cases} \pm 1 & \text{w.p. } \rho_0^{\pm}(x) \\ 0 & \text{w.p. } 1 - \rho_0^+(x) - \rho_0^-(x) \end{cases}$$

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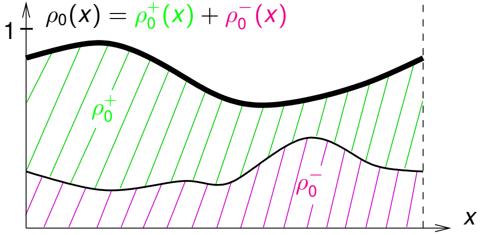

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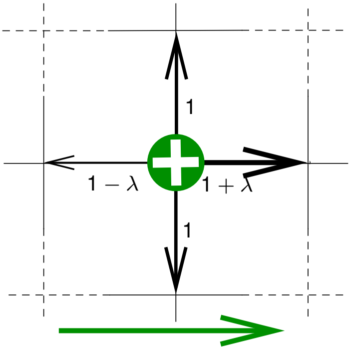
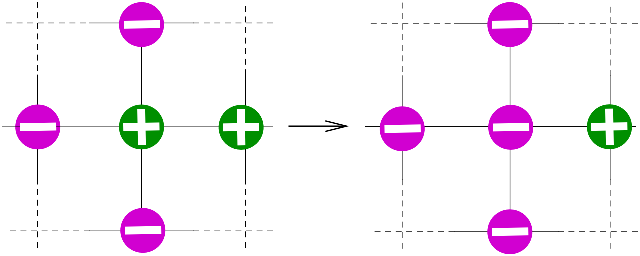
Weakly asymmetric exclusion



- ***Exclusion rule*** : if the target site is occupied, the motion is canceled
- $\lambda = \widehat{\lambda}/N$ is the strength of the ***weak asymmetry***

Glauber dynamics



$\mapsto$ *Ising type* alignment dynamics with inverse temperature β

- $\beta = 0$, no alignment
- $\beta \to \infty$, strong alignment

Generator of the dynamic

The Markov generator of the process is given by

$$L_N = N^2 \mathcal{L}^S + \widehat{\lambda} N \mathcal{L}^A + \mathcal{L}^G,$$

- $\mathcal{L}^S$: generator of the *Symmetric Simple Exclusion Process* (SSEP)

$$\mathcal{L}^S f(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{|z|=1} |\eta_x| \underbrace{(1 - |\eta_{x+z}|)}_{\text{Exclusion Rule}} \left(f(\eta^{x,x+z}) - f(\eta) \right).$$

- Diffusive scaling : $\times N^2$.

Generator of the dynamic

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- $\mathcal{L}^A$: generator of the *Asymmetric Simple Exclusion Process* (WASEP)

$$\mathcal{L}^A f(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{\delta = \pm 1} \delta \eta_x \underbrace{(1 - |\eta_{x+\delta e_1}|)}_{\text{Exclusion Rule}} \Big(f(\eta^{x, x+\delta e_1}) - f(\eta) \Big),$$

- Ballistic scaling : $\times N$.

Generator of the dynamic

$$L_N = N^2 \mathcal{L}^S + \widehat{\lambda} N \mathcal{L}^A + \mathcal{L}^G,$$

- $\mathcal{L}^G$: generator of the *Glauber alignment dynamics*

$$\mathcal{L}^G f(\eta) = \sum_{x \in \mathbb{T}_N} c_\beta(x, \eta) |\eta_x| \left(f(\eta^x) - f(\eta) \right).$$

- No need for rescaling.

The complete generator and the initial state define a Markov process $(\eta(t))_{t \in [0,T]}$.

Plan of the talk

Heuristic formulation of the macroscopic limit

Theorem

Assumption : $\forall u \in [0,1]^2$, $\rho_0(u) = \rho_0^+(u) + \rho_0^-(u) < 1$.

*The **macroscopic density** of particles $+$, denoted $\rho^+(t,u)$, is solution **in a weak sense** of the reaction-diffusion equation*

$$\partial_t \rho^+ = \nabla.\left[d_s(\rho)\nabla\rho^+ + D(\rho^+,\rho)\nabla\rho\right] + 2\widehat{\lambda}\partial_{x_1}\sigma(\rho^+,\rho) + \Gamma_\beta(\rho^+,\rho),$$

and

$$\rho^+(0,u) = \rho_0^+(u).$$

An analogous equation is verified by the $-$ particle density ρ^-, and $\rho = \rho^+ + \rho^-$ is the total particle density.

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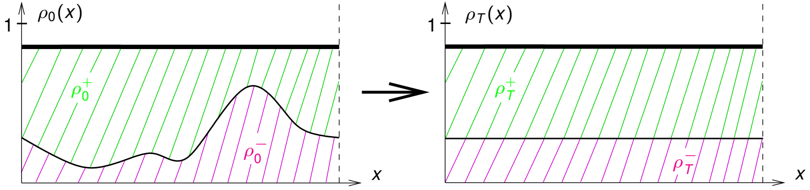
Dynamical interpretation

$$\partial_t \rho^+ = \nabla . \Big[d_s(\rho) \nabla \rho^+ + D(\rho^+, \rho) \nabla \rho \Big] + 2\widehat{\lambda} \partial_{x_1} \sigma(\rho^+, \rho) + \Gamma_\beta(\rho^+, \rho).$$



d_s is the ***self-diffusion coefficient*** of a tracer particle in an homogeneous environment.

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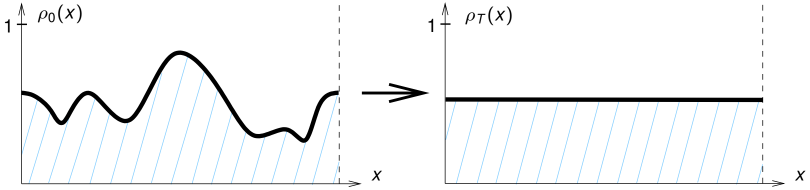

The coefficient D quantifies the **diffusion** due to heterogeneities of the **total particle density**

$$D(\rho^+, \rho) = \frac{\rho^+}{\rho}(1 - d_s(\rho)) \quad \text{(Quastel, 1992).}$$

Dynamical interpretation

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$\mapsto$ The drift σ can be expressed as

$$\sigma(\rho^+,\rho) = \frac{\rho^+}{\rho}(1-\rho-d_s(\rho))(\rho^+-\rho^-) + \rho^+ d_s(\rho),$$

and is linked to D and d_s by the Stokes-Einstein relation.

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- Γ_β is the creation rate of $''+''$ particles.
- It depends on the alignment jump rates c_β

Hydrodynamic limit

Empirical measures

$$\pi_t^{+,N} = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x^+(t)\delta_x, \quad \text{and} \quad \pi_t^{-,N} = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \eta_x^-(t)\delta_x.$$

We want to prove $\pi_t^{+,N} \xrightarrow{\mathbb{P}} \pi_t^+ = \rho^+(t,x)dx$, i.e. that for any smooth H,

$$< \pi_t^{+,N}, H > \to \int_{[0,1]^2} \rho^+(t,x)H(x)dx$$

Core principle :

$$< \pi_T^{+,N}, H > = < \pi_0^{+,N}, H > + \int_0^T L_N < \pi_t^{+,N}, H > dt + \overbrace{M_T^N}^{o_N(1)}$$

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Hydrodynamic limit

Assume we are in **one dimension** :

$$
\begin{aligned}
L_N < \pi_t^{+,N}, H > &= \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x) L_N \eta_x^+ \\
&= \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x) \left(W_{x-\frac{1}{N},x} - W_{x,x+\frac{1}{N}} + \gamma_{\beta,x} \right) \\
&= \frac{1}{N} \sum_{x \in \mathbb{T}_N} \left[\left(H(x) - H\left(x + \frac{1}{N} \right) \right) W_{x,x+\frac{1}{N}} + H(x)\gamma_{\beta,x} \right]
\end{aligned}
$$

and the particle current can be written

$$
W_{x,x+\frac{1}{N}} = N^2 \underbrace{w^S_{x,x+\frac{1}{N}}}_{\text{symmetric current}} + \widehat{\lambda} N \underbrace{w^A_{x,x+\frac{1}{N}}}_{\text{antisymmetric current}} .
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$$= \frac{1}{N} \sum_{x \in \mathbb{T}_N} \Bigg[\underbrace{\left(H(x) - H\left(x + \frac{1}{N} \right) \right)}_{\simeq \frac{1}{N} \partial_{u_i} H(x)} W_{x,x+\frac{1}{N}} + H(x) \gamma_{\beta,x} \Bigg]$$

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Non-gradient hydrodynamics

- The partial derivative on H balances out a factor N in the current.
- In ***non-gradient systems***, the symmetric current $w^S_{x,x+\frac{1}{N}}$ is not a discrete gradient.
  $\mapsto$ the second integration by parts is not immediate
- One must prove

$$w^S_{x,x+\frac{1}{N}} \simeq D(\eta_x - \eta_{x+1}) + d_s(\eta^+_x - \eta^+_{x+1}) + \mathcal{L}^S f$$

Out of equilibrium dynamics

Key issue

Comparison with a product measure on the discrete lattice :

- Distortion of the measure by the Glauber part and the initial configuration are easily controlled
- Distortion by the weak drift, harder to control

$\mapsto$ Challenge : prove that the exponential estimates needed in the *non-gradient* method still hold.

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Irreducibility

Exclusion rule : mixing is compromised in full clusters

- One must prove that the dynamics spreads enough empty sites to ensure mixing.
- This was a major issue with the model.
- The needed estimate is

$$\mathbb{E}\left(\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\mathbb{1}_{\text{Empty site in }\tau_x B_p}\right)\simeq\int_{[0,1]^2}dx\,\rho_x^{|B_p|}\underset{p\to\infty}{\longrightarrow}0. \quad (1)$$

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Continuous angle dynamic

Goal : extension of the model to a continuous type $\theta \in [0, 2\pi[$

Theorem, continuous angles

The macroscopic density ρ^θ of particles with angle θ is solution in a weak sense of

$$\partial_t \rho^\theta = \frac{1}{2}\nabla\left[d_s(\rho)\nabla\rho^\theta + D(\rho^\theta,\rho)\nabla\rho\right] + \widehat{\lambda}\nabla\sigma(\rho^\theta,\rho) + \Gamma^\theta$$

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- Γ^θ alignment term, continuous diffusion, jump process

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Thanks for your attention !