Ruin probability for a process with switching

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Abstract

We study various properties of a general subclass of real-valued Markov processes, namely the oscillating random processes, or random processes with several levels of switching. There is given a partition of the real line into a finite number of sets and, with each set, a probability distribution is associated. When a Markov chain has a value in one of these sets, its next increment has the distribution associated with that set.

Ruin probability is considered for a Markov process with one level of switching between two independent Lévy processes one of which is spectrally negative and another is a compound Poisson process with drift. Explicit representations were found for the ruin probability in terms of ladder heights. As a consequence, results were obtained for a risk process, where the premium rate and the claim size depend on whether current reserve is above or below a certain threshold.

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Content

1. Ruin probability for Lévy processes.
   - Two-boundary exit problem for spectrally negative Lévy process.
   - Explicit representations for Ruin probability.
2. Application. Ruin probability for Risk reserve process.

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Content

Let $\xi_i(t),\ \xi_i(0)=0,\ t\geq 0,\ i=1,2,$ be two independent random processes with stationary independent increments (Lévy processes) such that $\xi_1(t)$ is spectrally negative and $\xi_2(t)$ is a compound Poisson process with drift

$$\xi_2(t)=c_2t-\sum_{k=1}^{N_2(t)}U_k^{(2)},$$

where $c_2\geq 0,\ \{U_n^{(2)}\}_{n=1}^{\infty},$ is a sequence of i.i.d. random variables and $N_2(t)$ is a Poisson process with intensity $\mu_2>0$, and $\mathbf{P}\left(\sup_{t\geq 0}\xi_2(t)=\infty, \inf_{t\geq 0}\xi_2(t)>-\infty\right)=1.$

For $i = 1, 2,$ and an arbitrary real number x, define the random variables

$$\eta_-^{(i)}(x) = \inf\{t \geq 0 : \xi_i(t) < x\},$$
$$\eta_+^{(1)}(x) = \inf\{t \geq 0 : \xi_1(t) = x\},$$
$$\chi_-^{(i)}(x) = \xi_i\left(\eta_-^{(i)}(x)\right).$$

(we assume that $\inf \emptyset = \infty$ everywhere). For brevity, we further write $\chi_-^{(2)} = \chi_-^{(2)}(0)$, $\eta_-^{(2)} = \eta_-^{(2)}(0)$.

Let $b > 0$ and we define the oscillating random process with one switching level $X_y(t),\ t \geq 0,\ y \leq b$, as follows: $X_y(0) = y$ and put

$$X_y(t) = \begin{cases} y + \xi_1(t), \\ \quad \text{if} \quad 0 \leq t < \eta_+^{(1)}(b-y), \\ \\ b + \xi_2\left(t - \eta_+^{(1)}(b-y)\right), \\ \quad \text{if} \quad \eta_+^{(1)}(b-y) \leq t < \eta_+^{(1)}(b-y) + \eta_-^{(2)}, \\ \\ X_{b+\chi_2^-}(t), \\ \quad \text{if} \quad t \geq \eta_+^{(1)}(b-y) + \eta_-^{(2)}. \end{cases}$$

For brevity we posit $X(t) = X_0(t)$. Our aim is to consider the ruin probability

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right), \quad a < 0,$$

as a function of a and b.

Distribution of some functionals of oscillating processes were studied by **Yu. V. Prokhorov** (1964) and **E.V. Buliskaya** (1996), **D.V. Gusak, N.S. Bratijchuk, O.I. Eleyko** (1984), **V.I. Lotov** (1996), **V.I.Lotov and D.K. Kim** (2003, 2004, 2006).

Some results were given by **H.U. Gerber** (1975), **G.C. Taylor** (1980), **S. Asmussen** and **S.S. Petersen** (1988), **S.S. Petersen** (1990), **D.C.M Dickson** (1991), **H. Schmidli** (1994), **B.Sundt** and **J.L. Teugels** (1995), **F. De Vylder** (1996), **F. Michaud** (1996), **E. Marceau** (1999), **H. Jasiulewicz** (2001), **Wu Rong and Wei Li** (2004), **Lin** and **Pavlova** (2006), **Zhang** (2006), **A.E. Kyprianou, R.L. Loeffen** (2008).

They considered the risk reserve process $R(t),\ t \geq 0$,

$$R(t) = R(0) + \int_0^t c\,(R(s))\,ds - Y(t)$$

where $R(0)$ is the insurer's initial capital, $Y(t) \geq 0$ denotes aggregate claims up to time t, $c(\bullet)$ is the premium rate function. I know only a few results related to models where both premium rate and claim size depend on risk reserve of **M.S. Bratiychuk** and **D. Derfla** (2007), **R.Bekker, O.J. Boxma, O.Kella** (2008), **R.Bekker, O.J. Boxma, J.Resing** (2007), **R.Bekker O.J.Boxma** (2007), **O.J.Boxma, H.Johnsson, J. Resing, S. Schneer** (2008).

We also use the notations

$$\underline{\xi}_i = \inf_{t \geq 0} \xi_i(t), \quad \overline{\xi}_i = \sup_{t \geq 0} \xi_i(t),$$

$$\gamma_-^{(1)}(-x) = \chi_-^{(1)}(-x) - x, \quad x \geq 0,$$

$$p_i = \mathbf{P}\left(\eta_-^{(i)} < \infty\right), i = 1, 2.$$

Remark that $p_2 < 1$.

Now we consider a two-boundary exit problem for spectrally negative Lévy process, which we will reduce to a one-boundary and then observe the explicit representations for sought distribution.

Theorem 1 Let $\xi_1(t)$ be spectrally negative Lévy process, $a \le 0 \le b$.

1. If $\mathbf{E}\xi_1(1) > 0$ then

$$\mathbf{P}\left(\eta_-^{(1)}(a) < \eta_+^{(1)}(b)\right) = \frac{\mathbf{P}\left(\underline{\xi}_1 < a\right) - \mathbf{P}\left(\underline{\xi}_1 < a - b\right)}{1 - \mathbf{P}\left(\underline{\xi}_1 < a - b\right)}.$$

2. If $\mathbf{E}\xi_1(1) = 0$ and $\mathbf{E}\xi_1^2(1) < \infty$ then

$$\mathbf{P}\left(\eta_-^{(1)}(a) < \eta_+^{(1)}(b)\right) = \frac{b + \mathbf{E}\gamma_-^{(1)}(a) - \mathbf{E}\gamma_-^{(1)}(a-b)}{b - a - \mathbf{E}\gamma_-^{(1)}(a-b)}.$$

Theorem 1*(continuation)*

3. If $\mathbf{P}\left(\overline{\xi}_1 = \infty\right) = 0$ and $\mathbf{P}\left(\underline{\xi}_1 = -\infty\right) = 1$ then

$$\mathbf{P}\left(\eta_-^{(1)}(a) < \eta_+^{(1)}(b)\right) = 1 - e^{-qb} \frac{1 - e^{qa}\mathbf{E}e^{q\gamma_-^{(1)}(a)}}{1 - e^{q(a-b)}\mathbf{E}e^{q\gamma_-^{(1)}(a-b)}},$$

where $q > 0$ is the unique positive solution to $\mathbf{E}e^{q\xi_1(0)} = 1$.

Now, we are ready to find the explicit representations for $\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right)$, where the process $\xi_1(t)$ may have various drift.

Theorem 2

Let $\mathbf{E}\xi_1(1) > 0$. Then

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right) = \frac{\mathbf{P}\left(\underline{\xi}_1 < a\right) + D(a-b)}{1 + D(a-b)},$$

where

$$D(a-b) = \frac{\mathbf{P}\left(\underline{\xi}_1 + \chi_-^{(2)} < a-b, \eta_-^{(2)} < \infty\right) - \mathbf{P}\left(\underline{\xi}_1 < a-b\right)}{1-p_2}.$$

Theorem 3

Let $\mathbf{E}\xi_1(1) = 0$, $\mathbf{E}\xi_1^2(1) < \infty$. Then

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right) = \frac{b + \mathbf{E}\gamma_-^{(1)}(a) + M(a-b)}{b - a + M(a-b)},$$

where

$$M(a-b) = \frac{b-a}{1-p_2}\mathbf{P}\left(\chi_-^{(2)} < a - b, \eta_-^{(2)} < \infty\right) + \frac{L(a-b)}{1-p_2} - \mathbf{E}\gamma_-^{(1)}(a-b),$$

Theorem 3 *(continuation)*

where

$$L(a-b) = \mathbf{E}\left(\gamma_-^{(1)}\left(a-b-\chi_-^{(2)}\right); \chi_-^{(2)} \geq a-b, \eta_-^{(2)} < \infty\right)$$
$$- p_2 \mathbf{E}\gamma_-^{(1)}(a-b) - \mathbf{E}\left(\chi_-^{(2)}; \chi_-^{(2)} \geq a-b, \eta_-^{(2)} < \infty\right).$$

Theorem 4

Let $\mathbf{P}\left(\overline{\xi}_1 = \infty\right) = 0$ and $\mathbf{P}\left(\underline{\xi}_1 = -\infty\right) = 1$. Then

$$\mathbf{P}\left(\inf_{t \geq 0} X(t) < a\right) = 1 - e^{-qb} \frac{1 - p_2}{1 - R(a-b)} \frac{1 - e^{qa} \mathbf{E} e^{q\gamma_-^{(1)}(a)}}{1 - e^{-q(a-b)} \mathbf{E} e^{q\gamma_-^{(1)}(a-b)}},$$

where $q > 0$ is the unique positive solution to $\mathbf{E} e^{q\xi_1(0)} = 1$ and

Theorem 4 *(continuation)*

$$R(a-b) = \left(1 - e^{q(a-b)}\mathbf{E}\left(e^{q\gamma_-^{(1)}(a-b)}\right)\right)^{-1}$$
$$\times\left(\mathbf{E}\left(e^{q\chi_-^{(2)}}; \chi_-^{(2)} \geq a-b, \eta_-^{(2)} < \infty\right)\right.$$
$$\left.-e^{q(a-b)}\mathbf{E}\left(e^{q\gamma_-^{(1)}\left(a-b-\chi_-^{(2)}\right)}; \chi_-^{(2)} \geq a-b, \eta_-^{(2)} < \infty\right)\right).$$

The explicit representations in Theorems 2-4 allow us to study the asymptotics and obtain upper and lower bounds for the infinum of $X(t)$, $t \geq 0$, as $b - a \to \infty$, both for the "light-" and for the "heavy-" tailed distributions of the Lévy measure. For these asymptotics and bounds, the well-developed techniques can be applied. For instance, well-known asymptotic equivalences may be used for $\mathbf{P}\left(\inf_{t\geq 0} \xi_1(t) < y\right)$ and for $\mathbf{P}\left(\chi_-^{(2)} < y, \eta_-^{(2)} < \infty\right)$, as $y \to -\infty$.

Corollary 1

Let $\mathbf{E}\xi_1(1) = 0$, $\mathbf{E}|\xi_1(1)|^r < \infty$, $r > 3$ and $\mathbf{E}\xi_2(1) > 0$, $\mathbf{E}\xi_2^2(1) < \infty$. Then

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right)$$

$$\sim \frac{(1-p_2)b - \mathbf{E}\left(\chi_-^{(2)}, \eta_-^{(2)} < \infty\right)}{(1-p_2)\left(b - a - \frac{\mathbf{E}\left(\chi_-^{(1)}\right)^2}{2\mathbf{E}\chi_-^{(1)}}\right) - \mathbf{E}\left(\chi_-^{(2)}, \eta_-^{(2)} < \infty\right)},$$

$$b - a \to \infty.$$

Corollary 2

Let $\mathbf{E}|\xi_1(1)| < \infty$, $\mathbf{E}\xi_1(1) < 0$. Then

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right) \sim 1 - e^{-qb}\mathbf{E}e^{q\underline{\xi}_2}, \quad b - a \to \infty,$$

where $q > 0$ is the unique positive solution to $\mathbf{E}e^{q\xi_1(0)} = 1$.

The following problem in Risk Theory is considered. An insurance company, endowed with an initial capital a, receives premiums and pays out claims. A classical Risk reserve process of capital assets of an insurance company is given by the formula

$$X(t) = a + ct - \sum_{i=1}^{N(t)} U_i,$$

where $c > 0$ is a constant rate of income from the insurance premiums, U_i are claim sizes and $N(t)$ is a number of claims for the time $(0, t)$ (Poisson process). The main problem is to study the probability of the event that the capital is less than zero - Ruin probability.

Some natural extension examples of models not incorporated in the above set-up are:

1. $X(t) = a + ct + \sigma w(t) - \sum_{i=1}^{N(t)} U_i,$

where $\sigma > 0$ and $w(t)$ is Wiener process.

2. $X(t) = a + \xi(t),$

where $\xi(t)$ is spectrally negative Lévy process, i.e. random process with stationary independent increments without positive jumps. ≪ *Indeed very recent studies of problems to ruin in insurance risk has seen some preference to working with general spectrally negative Lévy process in place of the classical Risk processes.* ≫ [1]

[1] **A.E. Kyprianou and R.L. Loeffen** (2008)

3. Models with a premium, claim size and time between claims depending on the reserve $X(t)$.
The last item - models with a premium and claim size depending on the reserve $X(t)$ - can be applied in reinsurance.

Reinsurance means that the company (the *cedent*) insures a part of the risk at another insurance company (the *reinsurer*). Let $U \geq 0$ is a claim. A reinsurance arrangement is then defined in terms of function $h(x)$ with the property $h(x) \leq x$. Here $h(U)$ is the amount of the claim U to be paid by the *cedent* and $U - h(U)$ by the amount to be paid by the *reinsurer*.

The most common exmples are the following two:

1. **Stop-loss reinsurance** $h(U) = \min\{U, B\}$ for some $B > 0$.
2. **Proportional reinsurance** $h(U) = \theta U$ for some $\theta \in (0, 1)$. Also called *quota share* reinsurance.

We can consider Risk reserve process of capital assets of an insurance company, which reinsures its claims and pays for that from its premiums. For example,

$X(t) = a + c\,(X(t))\,t - \sum_{i=1}^{N(t)} \hat{U}_i\,(X(t))$, where claim size and the premium charge is assumed to depend on current reserve $X(t)$ as follows:

$$c(x) = \begin{cases} c_1, & \text{if } x \le b, \\ c_2, & \text{if } x > b. \end{cases}$$

$$\hat{U}_i(x) = \begin{cases} \min\{U_i, B\}, & \text{if } x \le b, \\ U_i \qquad\qquad\quad, & \text{if } x > b. \end{cases}$$

As an example of the application of Theorem 2 we will consider Ruin probability for two classical 'Risk reserve processes'

$$\xi_i(t) = c_i t - \sum_{k=1}^{N_i(t)} U_k^{(i)}, \quad i = 1, 2, \tag{1}$$

where $c_i > 0$ and $\{U_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, 2,$ are two independent sequences of positive i.i.d. random variables, $\mathbf{E}U_1^{(i)} < \infty$ and $N_i(t)$, $i = 1, 2$ are two indepenenet Poisson processes with intensities $\mu_i > 0$, which do not depend on $\{U_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, 2,$ and $\mathbf{E}\xi_i(1) = c_i - \mu_i \mathbf{E}U_1^{(i)} > 0$.

For these processes

$$\mathbf{P}\left(\chi_-^{(i)} < z, \eta_-^{(i)} < \infty\right) = \frac{\mu_i}{c_i}\int_{-z}^{\infty} \mathbf{P}\left(U_1^{(i)} > x\right) dz, \;\; z < 0,$$

and

$$p_i = \frac{\mu_i}{c_i}\mathbf{E} U_1^{(i)}, \;\; i = 1, 2.$$

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of positive i.i.d. random variables and T is a arbitrary positive number. Put

$$U_i^{(1)} \stackrel{d}{=} \min\{Y_i, T\}, \quad (2)$$

$$U_i^{(2)} \stackrel{d}{=} Y_i. \quad (3)$$

Under the condition (2) there exists $\delta = \delta(T) > 0$:

$$\mathbf{E} e^{\delta U_1^{(1)}} = 1 + \delta \frac{c_1}{\mu_1}.$$

Let

$$\overline{B}(x) = \mathbf{P}\left(Y_1 > x\right), \quad \overline{B}_0(y) = \frac{1}{\mathbf{E}Y_1}\int_y^\infty \overline{B}(x)dx.$$

Definition. We will say that $\overline{B}(x)$ is subexponential $(\overline{B} \in S)$ if

$$\frac{\overline{B^{*2}}(x)}{\overline{B}(x)} \to 2, \ x \to \infty.$$

Here $\overline{B^{*2}}$ is the convolution square.

Theorem 4 (Approximations)

Let $\xi_i(t)$, $i=1,2$, be two Risk reserve processes (1), $\mathbf{E}\xi_i(1)>0$ and conditions (2) and (3) are satisfied.

1. If $\mathbf{E}e^{\delta Y_1}<\infty$ and $a\to-\infty$ then

$$\mathbf{P}\left(\inf_{t\geq 0}X(t)<a\right)\sim c_1e^{\delta a}\frac{1-\frac{\mu_1}{c_1}\mathbf{E}\min(Y_1,T)}{\mu_1\delta\int_0^T xe^{\delta x}\overline{B}(x)dx}\times\left(1-e^{-\delta b}\kappa\frac{\mu_2}{c_2}\left(1-\frac{\mu_2}{c_2}\mathbf{E}Y_1\right)^{-1}\right),$$

where

$$\kappa = \frac{\mathbf{E}e^{\delta Y_1} - \left(1 + \delta \frac{c_2}{\mu_2}\right)}{\delta}$$

2. If $(\overline{B}_0 \in S)$ and $a \to -\infty$ then

$$\mathbf{P}\left(\inf_{t \geq 0} X(t) < a\right) \sim \frac{c_2 \mathbf{E} Y_1}{\mu_2 - c_2 \mathbf{E} Y_1} \overline{B}_0(b-a).$$

Under the condition (2) the following inequalities are well-known

$$\beta_1 e^{\delta z} \leq \mathbf{P}(\underline{\xi}_1 < z) \leq \beta_2 e^{\delta z},$$

where

$$\beta_1 = \inf_{x \geq 0} \frac{\mathbf{P}\left(U_1^{(1)} > x\right)}{\int_x^\infty e^{\delta(y-x)} \mathbf{P}\left(U_1^{(1)} \in dy\right)}$$

$$\beta_2 = \sup_{x \geq 0} \frac{\mathbf{P}\left(U_1^{(1)} > x\right)}{\int_x^\infty e^{\delta(y-x)} \mathbf{P}\left(U_1^{(1)} \in dy\right)},$$

On the basis of them we can otain the following upper and lower bounds for the sought Ruin probability.

Theorem 5 (Bounds)
Let $\xi_i(t)$, $i = 1, 2$, be two Risk reserve processes (1), $\mathbf{E}\xi_i(1) > 0$ and conditions (2) and (3) are satisfied. Then

$$\frac{(1-p_2)\beta_1 e^{\delta a} + p_2\overline{B}_0(b-a) + \beta_1 V(\delta) - \beta_2 e^{\delta(a-b)}}{1 - p_2 + p_2\overline{B}_0(b-a) + \beta_2 V(\delta) - \beta_1 e^{\delta(a-b)}}$$

$$\leq \mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right) \leq$$

$$\frac{(1-p_2)\beta_2 e^{\delta a} + p_2\overline{B}_0(b-a) + \beta_2 V(\delta) - \beta_1 e^{\delta(a-b)}}{1 - p_2 + p_2\overline{B}_0(b-a) + \beta_1 V(\delta) - \beta_2 e^{\delta(a-b)}},$$

where

$$V(\delta) = \frac{\mu_2}{c_2} e^{\delta(a-b)} \int_0^{b-a} e^{\delta x} \overline{B}(x) dx$$

and

$$p_2 = \frac{\mu_2}{c_2} \mathbf{E} Y_1.$$

Further we will assume that

$$U_1^{(1)} \stackrel{d}{=} \theta Y_1, \quad 0 < \theta < 1, \tag{4}$$

$$U_1^{(2)} \stackrel{d}{=} Y_1 \quad . \tag{5}$$

Definition. We will say that $\overline{B}_0(x)$ is regularly varying if

$$\overline{B}_0(x) \sim L(x)/x^{\alpha}, \quad \alpha > 0,$$

as $x \to \infty$, where $L(x)$ is slowly varying, $L(tx)/L(x) \to 1$, $x \to \infty$, for all $t > 0$.

Theorem 6

Let $\xi_i(t)$, $i = 1, 2$, be two Risk reserve processes (1), $\mathbf{E}\xi_i(1) > 0$ and conditions (4) and (5) are satisfied.

1. Approximation

If $\overline{B}_0(x)$ is regularly varying with parametre $\alpha > 0$, $\mathbf{E}Y_1^2 < \infty$ and $a \to -\infty$ then

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right) \sim \frac{\theta^{\alpha+1} c_1 c_2 \left(\mathbf{E}Y_1\right)^2}{\left(\mu_1 - c_1\theta\mathbf{E}Y_1\right)\left(\mu_2 - c_2\mathbf{E}Y_1\right)} \frac{L(-a)}{(-a)^{\alpha}}.$$

2. Bound

If $\mathbf{E}Y_1^2 < \infty$ and $b - a \geq d$ then

$$\mathbf{P}\left(\inf_{t\geq 0} X(t) < a\right) \leq \frac{d}{|a|} + (1-p_2)^{-1} \times \left(p_2\overline{B}_0(b-a-d) - \frac{p_1}{1-p_1}\overline{B}_0\left(\frac{b-a}{\theta}\right) + p_2 d\int_0^{b-a-d}(b-a-x)^{-1}\overline{B}(x)dx\right),$$

where

$$d = \theta^2 \frac{\mu_1}{2c_1} \left(1 - \theta \frac{\mu_1}{c_1} \mathbf{E} Y_1\right)^{-1},$$

$$\frac{p_1}{1 - p_1} = \theta \frac{\mu_1}{c_1} \mathbf{E} Y_1 \left(1 - \theta \frac{\mu_1}{c_1} \mathbf{E} Y_1\right)^{-1},$$

$$p_2 = \frac{\mu_2}{c_2} \mathbf{E} Y_1$$