Analytical Pricing of Basket Default Swaps

A dynamic model with auto-calibration to CDS curves

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Outline

1. **Model Setup**
   - Univariate Models (margins)
   - Multivariate Model (stochastic intensity)
2. **Main Analytical Results**
   - Survival and jointure functions
   - Application: Pricing of First-to-Default swaps
   - Playing with (H, λ)
   - Dealing with k^{th}-to-Default, $1 < k \leqslant N$
3. **Conclusion**

Context: Credit-Based Financial Instruments

- Valuation of financial products: price = $f(\vec{\tau})$, vector of $N \geqslant 1$ default times $\vec{\tau} \doteq (\tau_1, \ldots, \tau_N)$
- Example: CDS, CDO, FtD, NtD,. . .
- We need a tractable default model for the joint CDF F of $\vec{\tau}$

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- We need a tractable default model for the joint CDF F of $\vec{\tau}$
  - speed (computationally attractive)
  - flexible (calibration capabilities)
  - sparse (not too many parameters)
- Two-step (bottom-up) approach to create a multivariate model:
  1. model the univariate distributions $F_i(x) \doteq \Pr[\tau_i \leqslant t]$
  2. couple the F_i's to create the joint distribution F

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Default Model ($N = 1$): Standard set up

- Intensity process : $\lambda_i(t) > 0$ ($\forall t > 0$)
- Probabilities : let $\Lambda_i(t) \doteq \int_{s=0}^{t} \lambda_i(s) ds$, then

$$S_i(t) \doteq \Pr[\tau_i > t] = e^{-\int_{s=0}^{t} \lambda_i(s) ds} = e^{-\Lambda_i(t)}$$

$$F_i(t) \doteq \Pr[\tau_i \leqslant t] = 1 - S_i(t)$$

- Meaning :
  - $\lambda_i(t) \sim$ default rate @ t (= $\lim_{\Delta \to 0} \Pr[\tau_i \leqslant t + \Delta | \tau_i > t] / \Delta$)
  - $\lambda_i(t) \sim$ deterministic : piecewise constant bw tenors
  - $\tau_i \sim 1^{st}$ jump of Poisson process with intensity $\lambda_i(t)$

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Multivariate : N underlying entities are gathered in a portfolio

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Modeling Intensities : Hull & White [1/2]

- Link bw entities : $\Lambda_i(t)$ become stochastic: $\Lambda_i(t) \Rightarrow \tilde{\Lambda}_i(t)$
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- $S_i(t) = \Pr[P_i(t) > U_i]$, $U_1, \ldots, U_N$ are $\mathcal{U}(0, 1)$ rv

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Examples: ($U_i \perp U_j$ and $P_i(t) = e^{-\tilde{\Lambda}_i(t)}$)

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- $M_i(t) \doteq \int_{s=0}^{t} \mu_i(s)ds$ is a cumulative deterministic intensity
- J_t is a inhomogeneous Poisson process with intensity $\lambda(t)$
- $H(j) > 0$ defines size of j^{th} jump of $\sum_{j=1}^{J_t} H(j)$

Modeling Intensities : Hull & White [2/2]

- Define $\phi_X(u)$ the CF of X: $\phi_X(u) \doteq \mathbb{E}[e^{-iuX}]$
- Observe that $S_i(t) = \mathbb{E}[e^{-\tilde{\Lambda}_i(t)}] = \phi_{\tilde{\Lambda}_i(t)}(-i)$
- Calibration to CDS mkt :

$$\underbrace{\mathbb{E}[e^{-\tilde{\Lambda}_i(t)}]}_{S_i(t)\ \text{model}} = \underbrace{e^{-\Lambda_i(t)}}_{S_i(t)\ \text{CDS mkt}}$$

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$$\Updownarrow$$

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\end{aligned}
$$

- if $H(j) = H$, then ok in closed-form. Indeed:

Hull & White : Calibration to CDS

Survival probability of entity i as given by model:

$$\begin{aligned}
\mathbb{E}\left[e^{-M_i(t)-\sum_{j=1}^{J_t} H(j)}\right] &= e^{-M_i(t)}\mathbb{E}[e^{-J_t H}] \\
&= e^{-M_i(t)}\phi_{J_t}(iH) \\
&\overset{\Lambda(t)\doteq\int_{s=0}^{t}\lambda(s)ds}{=} e^{-M_i(t)}e^{\Lambda(t)(e^{-H}-1)}
\end{aligned}$$

So, calibration to CDS probs is achieved provided that

$$\mu_i(s) \overset{\forall s\leqslant t}{=} \lambda_i(s) - \lambda(s)(1-e^{-H})$$

Modeling probabilities vs Modeling events

- So far, we have required $\Pr[\tau_i > t] = \Pr[e^{-\tilde{\Lambda}_i(t)} > U_i]$
- This is not the same as requiring $\{\tau_i > t\} = \left\{e^{-\tilde{\Lambda}_i(t)} > U_i\right\}$
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- Therefore, we need the additional constraint $\mu_i(s) > 0$, which defines a range for $(\lambda(s), H)$.
- Condition $\mu_i(s) > 0$ is not needed to fit $S_i(t)$, but necessary to get proper conditional and joint distributions

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Hull & White : Jointure function

Let $\psi(N, H, \Lambda(t))$ be the jointure function :

$$\psi(N, H, \Lambda(t)) \doteq e^{\Lambda(t)\left((e^{-NH}-1)-N(e^{-H}-1)\right)}$$

It holds that

$$\psi(N, 0, \Lambda(t)) = \psi(N, H, 0) = 1$$

and if $N > 1, H > 0, \Lambda > 0$, then

$$\psi(N, H, \Lambda) > 1$$

Hull & White : Survival function

If $H(j) = H$ and $\lambda(t) = \lambda$:

$$
\begin{aligned}
S(t_1, \ldots, t_N) &\doteq \Pr[\tau_1 > t_1, \ldots, \tau_N > t_N] \\
&= \prod_{i=1}^{N} S_i(t_i)\psi\Big(N - i + 1, H, (t_{(i)} - t_{(i-1)})\lambda\Big) \\
S(\vec{t}) &= S^{\perp}(\vec{t}) \prod_{i=1}^{N} \psi\Big(N - i + 1, H, (t_{(i)} - t_{(i-1)})\lambda\Big)
\end{aligned}
$$

where $0 = t_{(0)} \leqslant t_{(1)} \leqslant \ldots \leqslant t_{(N)}$ is a permutation of $\{t_1, \ldots, t_N\}$

Hull & White : First to default distribution

Let $\tau^{(1)} \doteq \min_j \tau_j$:

$$\begin{aligned} S(t) &\doteq \Pr[\tau^{(1)} > t] \\ &= S(t, \ldots, t) \\ &= \psi\Big(N, H, \Lambda(t)\Big) S^{\perp}(t) \end{aligned}$$

where

$$S^{\perp}(t) \doteq \prod_{i=1}^{N} S_i(t)$$

Impact of jointure function ($N = 2$)

- Because $\psi \geqslant 1$: *PQD* $\Rightarrow \rho \geqslant 0$
- Bad news propagation effect:

$$\frac{\Pr[\tau_1 \leqslant x | \tau_2 \leqslant y]}{\Pr[\tau_1 \leqslant x]} = 1 + (\psi(2, H, \Lambda(t)) - 1) f(S_i(x), S_j(y))$$

- Pearson's correlation coefficient of $A_i(t) \doteq 1\!\!1_{\{\tau_i \leqslant t\}}$:

$$\mathrm{Corr}(A_i(t), A_j(t)) = \rho_{ij}(t) = (\psi(2, H, \Lambda(t)) - 1)\sqrt{f(S_i(t), S_j(t))}$$

- Short-term default correlation $\rho_{ij}(0) \doteq \lim_{t \downarrow 0} \rho_{ij}(t)$:

$$\rho_{ij}(0) = \frac{\log \psi(2, H, \lambda(0^+))}{\sqrt{\lambda_i(0^+)\lambda_j(0^+)}} \quad (\text{for GC} : \rho_{ij}(0) = 0)$$

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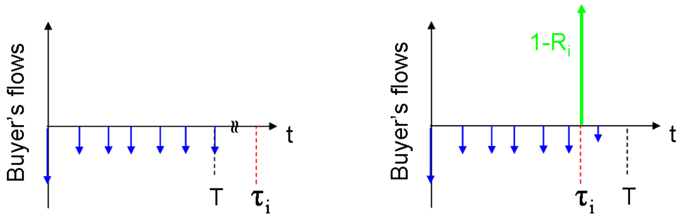
First-to-Default (FtD)

- Consider a basket of N companies and vector of default times $\vec{\tau}$ and a contract maturity time T
- Protection buyer pays upfront + premium up to $(T \wedge \min_i \tau_i)$
- Protection seller pays non-recovered part of notional of firstly defaulted entity iff $(\min_i \tau_i < T)$



Case : First-to-Default [1/3]

If all the entities have the same recovery rate $R_i = R$. Then

$$
\begin{aligned}
CL &\doteq \mathbb{E}\left[(1-R)\delta(\tau^{(1)})\,\mathbb{1}_{\{\tau^{(1)}\leqslant T\}}\right] = (1-R)\int_{t=0}^{T}\delta(t)f_{(1)}(t)dt \\
FL &\doteq s\sum_{k=1}^{K}\delta(t_k)\mathbb{E}\left[t_k \wedge (t_{k-1}\vee\tau^{(1)}) - t_{k-1}\right] \\
&= s\sum_{k=1}^{K}\delta(t_k)\left((t_k - t_{k-1})S(t_k) + \int_{t=t_{k-1}}^{t_k}(t-t_{k-1})f_{(1)}(t)dt\right)
\end{aligned}
$$

with $\delta(t)$ (disc. fact. at t), s (spread), $\{t_k\}$ (payment dates) and T (maturity)

Case : First-to-Default [2/3]

In terms of $S(t) \doteq \Pr[\tau^{(1)} > t]$, with $R_i = R$:

$$\begin{aligned} CL &\doteq -(1-R)\int_{t=0}^{T} \delta(t)dS(t) \\ FL &\doteq s\sum_{k=1}^{K} \delta(t_k)\int_{t=t_{k-1}}^{t_k} S(t)dt \end{aligned}$$

Hull & White : FtD priced as a CDS with intensity

$\Rightarrow$ If $R_i = R$, FtD = CDS :

$$S(t) = e^{-\tilde{\Lambda}(t)}, \quad \tilde{\Lambda}(t) \doteq \sum_{i=1}^{N} \Lambda_i(t) - \underbrace{\log \psi(N, H, \Lambda(t))}_{\doteq \lambda_0(t)}$$

$\Rightarrow$ FtD could be priced & calibrated with a HR-CDS pricer

$\Rightarrow$ FtD price range :

- $\overline{sp}$ (highest price): $\lambda_0(t) = 0$ (independence)
- $\underline{sp}$ (smallest price): $\lambda_0(t) = \sum_{i=1}^{N} \lambda_i(t)$? ($\tilde{\lambda} = 0 \Rightarrow sp = 0$)
- Actually, $\underline{sp} > 0$ as one must have $\lambda_0(t) < \sum_{i=1}^{N} \lambda_i(t)$

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Playing with (H, λ) = playing with tails [2/4]

Assume :

- $r(t) = 0$ (or equivalently, $\delta(t) = 1$, ie no interest rates)
- $S_i(t) = e^{-\lambda_i t}$ (one-tenor or one avg intensity up to maturity)
- $R_i = R$ (homogeneous recoveries)

Then, the iso-FtD curve $(H, \lambda(H, sp^\star))$ yielding a same fair spread $s^\star$ for FtD is given by

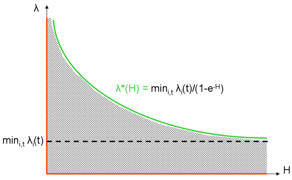
$$\lambda(H, s^\star) = \frac{\sum_{i=1}^{N} \lambda_i - \frac{sp^\star}{(1-R)}}{\log \psi(N, H, 1)}, \quad (H > 0)$$

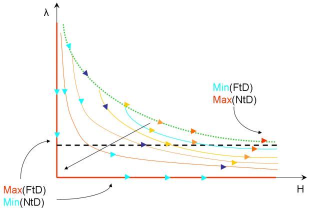
Indeed, in that case $\tilde{\lambda} = \lambda^\star$ where $\lambda^\star \doteq \frac{s^\star}{(1-R)}$ is the "fair intensity", ie the intensity such that CDS has a zero MtM when priced with $s^\star$ (when $r(t) = 0$).

$\Rightarrow$ Handy to calibrate KtD given FtD

Playing with (H, λ) = playing with tails [1/4]

- Couples (H, λ) fitting a same FtD price (= survival curve)





- Increase H s.t. FtD price is constant means
  - increase probability of catastrophe scenario
  - decrease implied jump intensity λ

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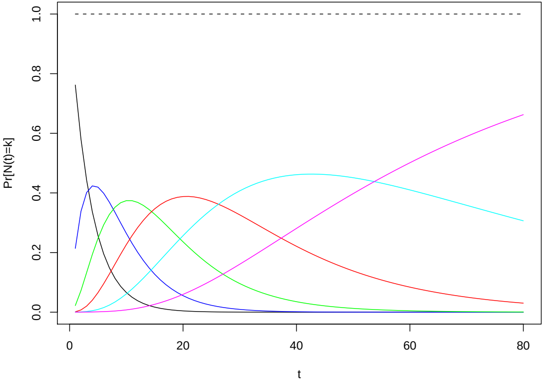
Hull & White : k^{th}-to-Default pricing [1/2]

- $\Pr[N(t) = k]$ is tractable for medium baskets
- Combinatorial but analytically tractable
- No approximation, no numerical integration, no recursion

$$\Pr\left[N(t) = k\right] = \sum_{\substack{1 \leqslant i_1 < \cdot < i_k \leqslant N \\ \{i_1, \cdot, i_k, j_1, \cdot, j_{N-k}\} = \{1, \cdot, N\}}} \prod_{k'=1}^{N-k} S_{j_{k'}}(t) \times \Bigg\{ \psi(N-k) + \sum_{l=1}^{k} (-1)^l \psi(N+l-k) \sum_{1 \leqslant m_1 < \cdot < m_l \leqslant k} \prod_{z=1}^{l} S_{i_{m_z}}(t) \Bigg\}$$

Hull & White : *k*th to default pricing [2/2]

- Example : N=5



Summary : Dynamic Models

- Standard copula models are static
- Dynamic copula difficult to work out (time-dependent barrier : no closed from solution for CDS calibration,. . .)
- Idea : Modeling multi-dimensional “intensity” processes with jumps to obtain sufficiently high default correlation
- Recent examples: Mai & Scherer, Hull & White

Summary : Jump models

- (+) Analytical results : not more difficult than static copula
- (+) Have the “fat tail effect”
- (+) Default correlations $\neq 0$ as $t \to 0$ (more stable)
- (+/-) Handling various R_i’s requires approximations due to simultaneous defaults

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