A queueing approach to a multi class $M/G/1$ make-to-stock with backlog

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Objectives

1. The equivalence of inventory model and queueing problem
2. Improving existing policies
M/G/1 make to stock

Single class

- A single machine produces a single item
- Demand process according to a $PP(\lambda)$
- i.i.d production times (G)
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- Decision: To produce or not. Taken at
  - production completion
  - arrival at idleness
$M/G/1$ make to stock

Stationary policy:
Produce $\iff$ the stock level is smaller than a base stock level $S$
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Optimal policy is stationary (MDP)
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What is the optimal $S$?
Equivalent queueing problem

An $M/G/1$ queue $(\lambda, G)$
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Optimization problem:

$$\min_{s \geq 0} hE(Q \mid Q < s) + bE(Q \mid Q > s)$$
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$$\min_{S \geq 0} h E(Q I\{Q < S\}) + b E(Q I\{Q > S\})$$

The optimal solution $S^*$ (Veatch and Wein 96):

$$P(Q \leq S^*) = \frac{b}{b + h}$$
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In this queue customers are served before their arrival!!
Multi class problem

- $N$ classes of customers
- arrival rates $\lambda_i$
- backlog costs $b_i$ ($b_i > b_{i+1}$)
Make to stock policies

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- **FCFS**
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  - Constants $S = R_{n+1} \geq R_n \ldots \geq R_1 \geq 0$
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$SP \subseteq IR \quad (R_i = 0)$
Some specific papers

- Ha (’97) Rationing $M/M/1$
- De Vericourt, Karasmen and Fallarey (’02) IR, FCFS, SP $M/M/1$
- J. P. Gayon, F. de Véricourt and F. Karaesmen (’07) IR $M/E_k/1$.
- Benjaafer, Elhafsi and Kim (05’) FCFS $M/G/1$
- Benjaafer, Elhafsi and Kim (07’) FCFS, IR $M/M/1$
- Abouee-Mehrizi, Balcioglu and Baron (’09) IR $M/G/1$

Markovian systems $\rightarrow$ Dynamic programming

Does not work for $M/G/1$
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Is IR an optimal control policy?

Optimal in $M/M/1$ (De Vericourt et al. ’02)
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At arrival epochs, allocate to class $i$ customer

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1. Inventory $> R_i$ or
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1. Inventory $> R_i$ \quad or

2. Inventory $= R_i$ and $\sum_{j=i}^{n} Q_j \geq q_i$}

\[ SP \subseteq IR \subseteq EIR \quad (q_i = \infty) \]
Analysis

Priority $M/G/1$ queue with state-dependent arrival rates.

$Q_1 = B_1 + S - I$

Embed at production completion epochs.

Balance equations....

Minimize

$$hE(Q_1 I_{Q_1 < S}) + b_1 E(Q_1 I_{Q_1 > S}) + \sum_{i=2}^{n} b_i E(Q_i)$$
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For example: If $G$ is DFR, the extension is:

If an arriving class $i$ customer finds $R_i + 1$,
allocate iff $\sum_{j=i+1}^{n} Q_j$ is small.
QUESTIONS?