First passage percolation on random graphs

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Joint work with:
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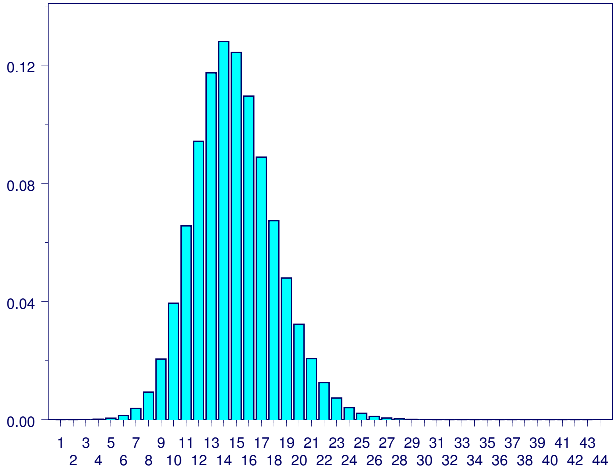
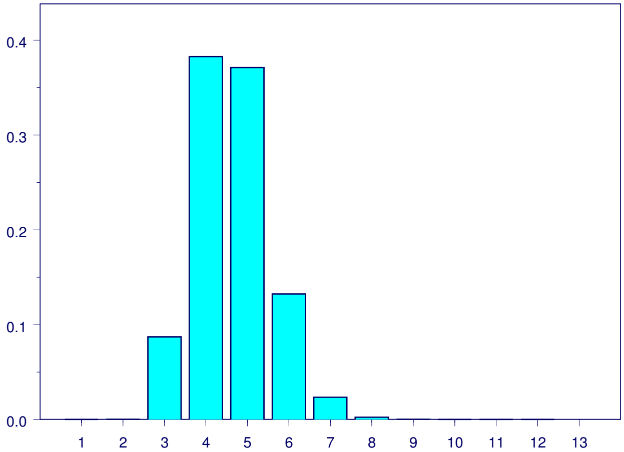
- Dmitri Znamenski (EURANDOM, now Philips Research)

Distances in IP graph



Poisson distribution??

Small-world phenomenon in AS graph



Distances in AS graph: Six degrees of separation?

Shortest-weight problems

In many applications, edge weights represent cost structure of the graph, such as actual economic costs or congestion costs across edges.

Actual time delay experienced by vertices in the network is given by hop-count H_n which is the number of edges on shortest-weight path.

How does weight structure influence hopcount and weight SWP?

Assume that

edge weights are i.i.d. (standard) exponential random variables.

Problem has received tremendous attention on $\mathbb{Z}^d$ and on complete graph. Now extend to first passage percolation on random graphs, in similar setting as in Adrea Montanari's talks.

Configuration model

Let n be the number of vertices. Consider i.i.d. sequence of degrees $D_1, D_2, \ldots, D_n$ with a certain distribution.

Special attention for power-law degrees, i.e., when

$$\mathbb{P}(D_1 \geq k) = c_\tau k^{-\tau+1}(1 + o(1)),$$

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When $\tau > 3$, we only need the above upper bound, giving us more flexibility in the choice of degrees.

Configuration model: graph construction

How to construct graph with above degree sequence?

- Assign to vertex j degree D_j.

$$L_n = \sum_{i=1}^{n} D_i$$

is total degree. Assume L_n is even.
Incident to vertex i have D_i 'stubs' or half edges.

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• Connect stubs to create edges as follows:
Number stubs from 1 to L_n in any order.
First connect first stub at random with one of *other* $L_n - 1$ stubs.
Continue with second stub (when not connected to first) and so on, until all stubs are connected...

Results

Theorem 1. (BHH09a-b). Let H_n be number of edges between two uniformly chosen vertices.
Assume $D \geq 2$ a.s. and $\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} > 1$.

For $\tau > 3$ or $\tau \in (2,3)$,

$$\frac{H_n - \alpha \log n}{\sqrt{\alpha \log n}} \xrightarrow{d} Z,$$

where Z is standard normal, and

$$\alpha = \frac{\nu}{\nu - 1} > 1 \qquad \text{for} \quad \tau > 3,$$
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When $\tau \in [1, 2)$, then

$$H_n \quad \text{uniformly bounded.}$$

Results

Theorem 2. (BHH09a-b). Let W_n be weight of shortest path between two uniformly chosen vertices.
Assume $D \geq 2$ a.s. and $\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} > 1$.

Then, for some limiting random variable W, and for $\tau > 3$ or $\tau \in (2,3)$,

$$W_n - \gamma \log n \xrightarrow{d} W,$$

where

$$\begin{aligned} \gamma &= \frac{1}{\nu - 1} > 0 && \text{for} \quad \tau > 3, \\ \gamma &= 0 && \text{for} \quad \tau \in (2,3). \end{aligned}$$

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- When $\tau \in (1, 2)$, (EHHZ06)

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Conclusion

- Random weights have marked effect on shortest-weight problem.

- Surprisingly universal behavior for FPP on configuration model.
Implications Internet hopcount?

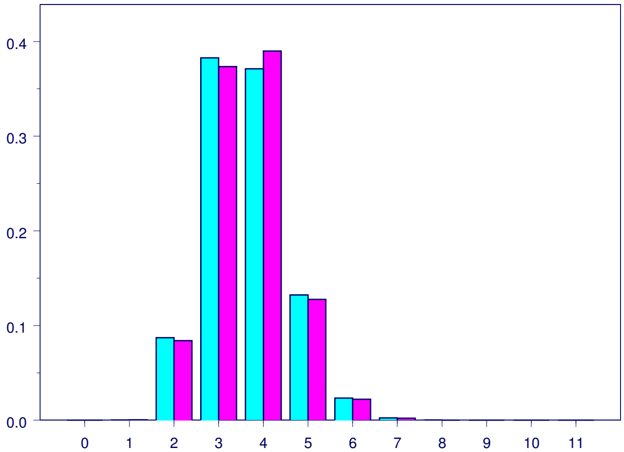
- Universality is leading paradigm in statistical physics.
Only few examples where universality can be rigorously proved.

Key question:
To what extent is universality true for random graphs models?

- More information Erdős-Rényi + power-law degree random graphs:

`www.win.tue.nl/∼rhofstad/NotesRGCN.pdf`

Comparison Internet data



Number of AS traversed in hopcount data (blue) compared to the model (purple) with $\tau = 2.25, n = 10,940.$