Estimation of the Degree of Jump Activity for a Lévy Process

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Eindhoven, 16 July 2009

Problem Formulation

- An underlying Lévy process $(X_t)_{t \geq 0}$
- Observations at equally spaced discrete times:

$$X_0, X_\Delta, \ldots, X_{n\Delta},$$

where $\Delta > 0$ is fixed.

Problem

Determine the "degree of activity" of the jumps, when the number of observations $n \to \infty$:

- *finitely many jumps ?*
- *or, if infinitely many, how "infinite" is it ?*

Exponential Lévy Models

$$S_t = S_0 \exp(rt + X_t),$$

where X_t is a Lévy process, i.e.

- $X_u - X_t$ is independent of $\mathcal{F}_t$ for $u > t$,
- $X_u - X_t$ is distributed as X_{u-t} for $u > t$,
- for any $\varepsilon > 0$,

$$\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0.$$

Lévy-Khintchine Formula

It holds for $\psi(u) := t^{-1}\log(\phi_t(u))$, where $\phi_t(u) = \mathrm{E}\exp(iuX_t)$

$$\psi(u) = -\frac{\sigma^2}{2}u^2 + i\gamma u + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\le 1\}})\nu(dx),$$

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(dx) < \infty,$$

$$\mathcal{T} := (\sigma, \gamma, \nu) \text{ – Lévy triple of } X: \ X \sim \mathcal{L}(\sigma, \gamma, \nu).$$

Jump Activity Index of a Lévy Process

Let X_t be a Lévy process with the Lévy measure ν. The value

$$\alpha = \inf\left\{ r \geq 0 : \int_{|x|\leq 1} |x|^r \nu(dx) < \infty \right\}$$

is called the **degree of jump activity** or the **fractional order** of the Lévy process X_t

- if X_t is a stable process, then α is the stability index of X_t,
- if X_t has finite activity, i.e. $\nu[-\varepsilon, \varepsilon] < \infty$, then $\alpha = 0$,
- if X_t has infinite activity and $\nu((-\infty, -\varepsilon] \cup [\epsilon, \infty)) \asymp \pi(\varepsilon)|\epsilon|^{-\alpha}$.

Lévy Processes with Asymptotically Stable Behavior

Lévy processes with **asymptotically α-stable behavior** (AS)

$$\psi(u) = i\gamma u - \sigma^2 u^2/2 + \vartheta(u), \quad \vartheta(u) = -\eta|u|^\alpha \tau(u), \quad u \in \mathbb{R},$$

where

- $$0 < \alpha < 2 - \text{index of AS}, \quad \operatorname{Re}\eta > 0,$$
- $$\operatorname{Re}\tau(u) > 0, \quad |u| > 0 \quad \text{and} \quad \tau(u) \to 1, \quad |u| \to \infty.$$

Lévy Processes with Asymptotically Stable Behavior

Let $\nu(x)$ be the Lévy density of AS Lévy process with

$$\tau(u) = 1 + D_{\pm} u^{-\kappa} + o(|u|^{-\kappa}), \quad u \to \pm\infty.$$

Then

$$\int_{|x|<\varepsilon} x^2 \nu(x)\, dx = c\varepsilon^{2-\alpha}\theta(\varepsilon),$$

where $c > 0$ is a constant depending on α and the function $\theta(\varepsilon)$ satisfies

$$|\theta(\varepsilon) - 1| \lesssim |\varepsilon|^{\kappa}, \quad \varepsilon \to 0.$$

Examples

Normal Inverse Gaussian (Barndorff-Nielsen)($\alpha = 1$)

$$\vartheta(u) = c\left[(\theta^2 - (\beta + iu)^2)^{1/2} - (\theta^2 - \beta^2)^{1/2}\right], \quad \theta > |\beta| > 0$$

Hyperbolic (Eberlein)($\alpha = 1$)

$$\phi(u) = \frac{\theta}{K_1(\theta\delta)} \frac{K_1(\delta\sqrt{\theta^2 + u^2})}{\sqrt{\theta^2 + u^2}}, \quad \theta, \delta > 0$$

Tempered stable (Carr & Madan)

$$\vartheta(u) = c\Gamma(-\alpha)[\lambda_+^\alpha - (\lambda_+ + iu)^\alpha + (-\lambda_-)^\alpha - (-\lambda_- - iu)^\alpha]$$

Asset Dynamics

We assume that the asset price S_t follows an exponential Lévy model under both historical measure $\mathbb{P}$ and the risk neutral measure $\mathbb{Q}$:

$$S_t = \begin{cases} Se^{X_t}, & \text{under } \mathbb{P} \\ Se^{rt+Y_t}, & \text{under } \mathbb{Q}. \end{cases}$$

$\mathbb{P} \sim \mathbb{Q}$ implies

$$\int_{\mathbb{R}} (\sqrt{d\nu^{\mathbb{Q}}/d\nu^{\mathbb{P}}} - 1)^2 \nu^{\mathbb{P}}(dx) < \infty.$$

Corollary

If X_t and Y_t are two AS Lévy processes and X_t has index α, then Y_t has the same index α.

Main Idea

Assume first that $\phi(u) = \exp(\psi(u))$ is known exactly. Define

$$\mathcal{L}(u) := \log(|\phi(u)|^2) = -\sigma^2 u^2 + 2\,\mathrm{Re}[\vartheta(u)]$$

and

$$\mathcal{L}_\xi(u) := \xi^2 \mathcal{L}(u) - \mathcal{L}(\xi u) = \log\left(|\phi(u)|^{2\xi^2} / |\phi(\xi u)|^2\right) =: \log(\varrho_\xi(u))$$

for some $\xi > 1$. Then it holds

$$\mathcal{L}_\xi(u) = -2\eta|u|^\alpha \left(\xi^2\,\mathrm{Re}[\tau(u)] - \xi^\alpha\,\mathrm{Re}[\tau(\xi u)]\right) = -c_\xi(\alpha)|u|^\alpha \tau_\xi(u),$$

where $c_\xi(\alpha) = 2\eta(\xi^2 - \xi^\alpha)^{-1}$ and $\tau_\xi(u) \to 1$.

Truncation

Let $\widetilde{\phi}(u)$ be an estimate of $\phi(u)$. Define

$$\mathcal{Y}_\xi(u) := \log(-\log(\varrho_\xi(u))) = \log(c_\xi) + \alpha\log(u) + \log(\mathrm{Re}\,\tau_\xi(u)),$$

$$\widetilde{\mathcal{Y}}_\xi(u) := \log(-\log(T_{\omega_-,\omega_+}[\widetilde{\varrho}_\xi](u))), \quad u > 0,$$

where

$$\widetilde{\varrho}_\xi(u) = |\widetilde{\phi}(u)|^{2\xi^2}/|\widetilde{\phi}(\xi u)|^2$$

and T_{ω_-,ω_+} is truncation operator with truncation levels $0 < \omega_- < \omega_+ < 1$

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Spectral Cut-Off Estimation

Let $w^1(u)$ be a function supported on $[\epsilon, 1]$ for some $\epsilon > 0$ that satisfies

$$\int_0^1 w^1(u)\log(u)\,du = 1, \quad \int_0^1 w^1(u)\,du = 0$$

Set $w^U(u) = U^{-1} w^1(uU^{-1})$ and define

$$\widetilde{\alpha}_{\xi,U} = \int_0^\infty w^U(u)\widetilde{\mathcal{Y}}_\xi(u)\,du.$$

Spectral Cut-Off Estimation

If $\widetilde{\mathcal{Y}}_\xi = \mathcal{Y}_\xi$

$$\widetilde{\alpha}_{\xi,U} = \log(c_\xi) \underbrace{\int_0^\infty w^U(u)\, du}_{0} + \alpha \underbrace{\int_0^\infty w^U(u) \log(u)\, du}_{1} + R_U,$$

where

$$R_U = \int_0^\infty w^U(u) \log(\operatorname{Re} \tau_\xi(u))\, du - \text{“modelling bias”}.$$

Minimax Model Class

Let us consider the class of AS Lévy models $\mathcal{A}(\bar{\sigma}, \bar{\alpha}, \eta_-, \eta_+, \varkappa)$ with

$$\psi(u) = \mathrm{i}\mu u - \sigma^2 u^2/2 + \vartheta(u), \quad \vartheta(u) = -\eta |u|^\alpha \tau(u), \quad u \in \mathbb{R},$$

where $0 < \alpha \le \bar{\alpha} < 2$, $0 \le \sigma \le \bar{\sigma}$,

$$0 < \eta_- \le |\eta| \le \eta_+ < \infty,$$

$$|1 - \tau(u)| \lesssim \frac{1}{|u|^\varkappa}, \quad |u| \to \infty.$$

We shall write

$$(\sigma, \alpha, \eta, \tau) \in \mathcal{A}(\bar{\sigma}, \bar{\alpha}, \eta_-, \eta_+, \varkappa).$$

Estimation of ϕ

Assume that the values of the log-price process $X_t = \log(S_t)$ on the equidistant time grid $\pi = \{t_0, t_1, \ldots, t_n\}$ are observed.

- Estimate the characteristic function $\phi(u)$ by its empirical counterpart

$$\widetilde{\phi}(u) = \frac{1}{n} \sum_{j=1}^{n} e^{\mathrm{i}u(X_{t_j} - X_{t_{j-1}})}.$$

Upper Bounds

For $U = \bar{U}$ with

$$\bar{U} = \left[\frac{1}{2\bar{\sigma}} \log(n \log^{-\beta}(n))\right]^{1/2}, \quad \beta > 0$$

it holds

$$\sup_{(\sigma,\alpha,\eta,\varkappa)\in\mathcal{A}(\bar{\sigma},\bar{\alpha},\eta_-,\eta_+,\varkappa)} \mathrm{E}\left|\widetilde{\alpha}_{\xi,\bar{U}} - \alpha\right|^2 \lesssim \mathcal{R}_n,$$

where

$$\mathcal{R}_n = c_\xi^{-1}(\alpha)\left[\frac{1}{2\bar{\sigma}} \log(n)\right]^{-\varkappa}.$$

Lower Bounds

For any fixed $\varkappa > 0$

$$\liminf_{n\to\infty} \inf_{\widetilde{\alpha}} \sup_{(\sigma,\alpha,\eta,\tau)\in\mathcal{A}(\bar{\sigma},\bar{\alpha},\eta_-,\eta_+,\varkappa)} \delta_n^{-2}\,\mathrm{E}(|\widetilde{\alpha}-\alpha|^2) = O(1),$$

where

$$\delta_n = \left[\frac{1}{2\bar{\sigma}}\log(n)\right]^{-\varkappa/2},$$

and the infimum is taken over all estimators $\widetilde{\alpha}$ of α.

Adaptive Procedure

Fix a sequence of cut-off parameters $U_1 > U_2 > \ldots > U_K$ and define

$$\widetilde{\alpha}_k = \int_0^\infty w^{U_k}(u)\widetilde{\mathcal{Y}}_\xi(u)\,du, \quad k = 1, \ldots, K.$$

For the sequence of estimates $\widetilde{\alpha}_k$ consider the sequence of nested hypothesis $H_k : \alpha_1 = \ldots = \alpha_k = \alpha$ with

$$\alpha_k = \int_0^\infty w^{U_k}(u)\mathcal{Y}_\xi(u)\,du, \quad k = 1, \ldots, K.$$

Adaptive Procedure

Compute sequentially

$$\widehat{\alpha}_k = \gamma_k \widetilde{\alpha}_k + (1 - \gamma_k)\widehat{\alpha}_{k-1},$$

where $\widehat{\alpha}_1 = \widetilde{\alpha}_1$ and the mixing parameter γ_k is defined as

$$\gamma_k := \mathcal{K}(T_k/\mathcal{V}_k), \quad T_k := \left(\widetilde{\alpha}_k - \widehat{\alpha}_{k-1}\right)^2 / v_k$$

- v_k is the variance of $\widetilde{\alpha}_k$
- $\mathcal{K}$ is a kernel supported on $[0, 1]$
- $\{\mathcal{V}_k\}$ is a set of critical values

Choice of Critical Values

The critical values $\mathcal{V}_1, \ldots, \mathcal{V}_{K-1}$ are selected to provide the prescribed performance under the simplest (null) hypothesis

$$\alpha_1 = \ldots = \alpha_K = \alpha.$$

Under the null hypothesis, every estimate $\widetilde{\alpha}_k$ fulfills asymptotically

$$\sqrt{n}(\widetilde{\alpha}_k - \alpha) \sim \mathcal{N}(0, v_k), \quad n \to \infty.$$

and therefore

$$\mathrm{E}_0 \left| v_{k,n}^{-1} \left(\widetilde{\alpha}_k - \alpha \right)^2 \right|^r \approx C_r,$$

where $v_{k,n} = n^{-1} v_k$, $C_r = E|\xi|^{2r}$ and ξ is standard normal.

Choice of Critical Values

The critical values $\mathcal{V}_1, \ldots, \mathcal{V}_{K-1}$ are selected in such a way that

$$\mathrm{E}_0 \left| v_{k,n}^{-1} \left(\widehat{\alpha}_k - \widetilde{\alpha}_k \right)^2 \right|^r \leq \gamma C_r \,, \qquad k = 2, \ldots, K,$$

where γ is the preselected confidence level of the procedure.

Numerical Example

Consider the GH Lévy model with parameters $(\kappa, \beta, \delta, \lambda)$ and c.f.

$$\Phi_{GH}(u) = e^{\mathrm{i}\mu u - \sigma^2 u^2/2} \frac{\left(\sqrt{\kappa^2 + \beta^2}\right)^{\lambda}}{\left(\sqrt{\kappa^2 - (\beta + \mathrm{i}u)^2}\right)^{\lambda}} \frac{K_\lambda\left(\delta\sqrt{\kappa^2 - (\beta + \mathrm{i}u)^2}\right)}{K_\lambda\left(\delta\sqrt{\kappa^2 + \beta^2}\right)}.$$

Lévy-Khintchine representation

$$\Phi_{GH}(u) = \exp\left(\mathrm{i}bu - \sigma^2 u^2/2 + \int_{-\infty}^{\infty} (e^{\mathrm{i}ux} - 1 - \mathrm{i}ux) g(x)\, dx\right),$$

where $g(x) = \frac{\delta}{\pi} x^{-2} + \frac{\lambda + \frac{1}{2}}{2} |x|^{-1} + \frac{\delta\beta}{\pi} x^{-1} + o(|x|^{-1}), \quad x \to 0.$

Numerical Example

- Simulate 500 samples of $(X_0, X_\Delta, \ldots, X_{n\Delta})$ from the GH model with $\lambda = 1$, $\beta = 0$, $\kappa = 1$ and $\delta = 4$.
- Construct

$$\widetilde{\phi}(u) = \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(X_{k\Delta} - X_{(k-1)\Delta})}.$$

- Compute

$$\widetilde{\mathcal{Y}}_\xi(u) := \log(-\log(T_{\omega_-,\omega_+}[\widetilde{\varrho}_\xi](u))),$$

where $\widetilde{\varrho}_\xi(u) = |\widetilde{\phi}(u)|^{2\xi^2}/|\widetilde{\phi}(\xi u)|^2$ and $\xi = 2$.

Numerical Example

- Solve the minimization problem

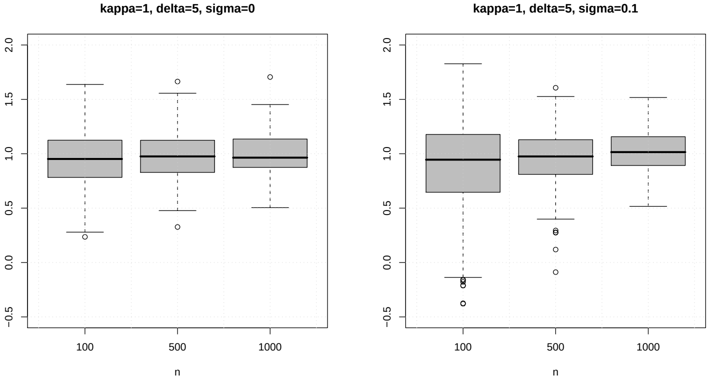
$$(l_0^U, l_1^U) = \operatorname{argmin}_{l_0, l_1} \int_0^U w^U(u)(\widetilde{\mathcal{Y}}_\xi(u) - l_1 \log(u) - l_0)^2 \, du,$$

where $w^U(u) = U^{-1} w^1(U^{-1}u)$ and $w^1(u) = u\mathbf{1}_{\{0 \le u \le 1\}}$.

- Let $U_k = 100(1.25)^{-(k-1)}, \quad k = 1, \ldots, 30$ and define $\widetilde{\alpha}_k = l_k^U$.

- Construct aggregated estimates $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{30}$ via SA with $\mathcal{K}(x) = (1 - x)1_{\{0 \le x \le 1\}}$, $r = 1$ and $\gamma = 0.5$.

Numerical Example



Box plots of the estimates $\widehat{\alpha}$ (left) and $\widehat{\alpha}_\xi$ (right) for different sample sizes.

Future Work

- Extension to the affine models
- Extension to time the changed Lévy models
- Investigate the case $\Delta \to 0$

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