STAR
APPLIED PROBABILITY FROM A QUEUEING PERSPECTIVE

Michel Mandjes

Korteweg-de Vries Institute for Mathematics, University of Amsterdam, the Netherlands &
CWI, Amsterdam, the Netherlands &
Eurandom, Eindhoven, the Netherlands

STAR

STAR

Intensifying an existing collaboration...

STAR

Stochastic Operations Research

STAR

Stochastic Operations Research:

how to optimally allocate scarce resources in an inherently uncertain environment.

Examples:

- ⋆ Network design;
- ⋆ Logistic and manufacturing networks;
- ⋆ Production systems;
- ⋆ Health care systems;
- ⋆ . . .

STAR

Stochastic Operations Research:

how to optimally allocate scarce resources in an inherently uncertain environment.

Examples:

- ⋆ Network design;
- ⋆ Logistic and manufacturing networks;
- ⋆ Production systems;
- ⋆ Health care systems;
- ⋆ . . .

Central notion: queue.

STAR

More recent insights:

- ⋆ Queueing theory can be applied when analyzing ruin probabilities in insurance risk;
- ⋆ Stochastic modelling is a promising technique in (system) biology;
- ⋆ . . .

STAR

Stochastic Operations Research: traditionally strong in the Netherlands.

Groups in Amsterdam (VU, CWI, UvA), Eindhoven, Twente, ...

STAR

Stochastic Operations Research: traditionally strong in the Netherlands.

Groups in Amsterdam (VU, CWI, UvA), Eindhoven, Twente, ...

Also internationally central position.

Major conferences: INFORMS Appl. Prob. (Eindhoven 2007), ITC (Amsterdam 2010), NETCOOP (Eindhoven 2009), STOPERA (Amsterdam 2006).

SIMULATION-BASED COMPUTATION OF THE WORKLOAD CORRELATION FUNCTION IN A LÉVY-DRIVEN QUEUE

Michel Mandjes

Korteweg-de Vries Institute for Mathematics, University of Amsterdam, the Netherlands &
CWI, Amsterdam, the Netherlands &
Eurandom, Eindhoven, the Netherlands

STAR — Cluster in Stochastics

Joint work with Peter Glynn (Stanford)

WHAT IS A QUEUE?

WHAT IS A QUEUE?

I. Definition as extension of ‘natural notion’ in discrete time.

Let $Q_0 = x$. Then

$$Q_{n+1} = \max\{Q_n + Y_n, 0\}.$$

WHAT IS A QUEUE?

I. Definition as extension of 'natural notion' in discrete time.

Let $Q_0 = x$. Then

$$Q_{n+1} = \max\{Q_n + Y_n, 0\}.$$

Iterate: $Q_{n+1} = \max\{Q_{n-1} + Y_{n-1} + Y_n, Y_n, 0\}$. With $X_n := \sum_{i=0}^{n} Y_i$, this leads to

$$Q_n = X_n + \max\left\{x, \max_{0 \le i \le n} -X_i\right\}.$$

WHAT IS A QUEUE?

I. Definition as extension of 'natural notion' in discrete time.

Let $Q_0 = x$. Then

$$Q_{n+1} = \max\{Q_n + Y_n, 0\}.$$

Iterate: $Q_{n+1} = \max\{Q_{n-1} + Y_{n-1} + Y_n, Y_n, 0\}$. With $X_n := \sum_{i=0}^{n} Y_i$, this leads to

$$Q_n = X_n + \max\left\{x, \max_{0\le i\le n} -X_i\right\}.$$

Then take the continuous-time counterpart:

$$Q_t = X_t + \max\{x, L_t\},\ t \ge 0,$$

with

$$L_t := \sup_{0\le u\le t} -X_u = -\inf_{0\le u\le t} X_u.$$

WHAT IS A QUEUE?

II. Definition through Skorokhod problem: $(Q_t)_t$ is reflection of $(X_t)_t$ at 0.

Let $(L^\star_t)_t$ be a nondecreasing right-continuous process such that

(a) $(Q_t)_t$, given by $Q_0 = x$ and $Q_t = X_t + L^\star_t$, is non-negative for all $t \geq 0$;

(b) $L^\star_t$ can only increase when $Q_t = 0$, that is

$$\int_0^T Q_t \mathrm{d}L^\star_t = 0, \quad \text{for all } T > 0.$$

WHAT IS A QUEUE?

II. Definition through Skorokhod problem: $(Q_t)_t$ is reflection of $(X_t)_t$ at 0.

Let $(L^\star_t)_t$ be a nondecreasing right-continuous process such that

(a) $(Q_t)_t$, given by $Q_0 = x$ and $Q_t = X_t + L^\star_t$, is non-negative for all $t \geq 0$;

(b) $L^\star_t$ can only increase when $Q_t = 0$, that is

$$\int_0^T Q_t \mathrm{d}L^\star_t = 0, \quad \text{for all } T > 0.$$

Then

$$Q_t = X_t + L^\star_t = X_t + \max\{x, L_t\},\ t \geq 0,$$

and (local time!)

$$L_t = -M_t := -\inf_{0 \leq u \leq t} X_u.$$

WHAT IS A QUEUE?

In this way we have a well-defined notion of ‘a queue fed by Brownian motion’

(also called ‘reflected Brownian motion’).

WHAT IS A QUEUE?

In this way we have a well-defined notion of ‘a queue fed by Brownian motion’

(also called ‘reflected Brownian motion’).

Queue with stationary increments: $X_t - X_{t-s}$ has, for a given s, the same distribution, irrespective of t.

Steady-state distribution $Q \equiv \lim_{t\to\infty} Q_t$ exists if $\mathbb{E}X_1 < 0$.

WHAT IS A QUEUE?

In this way we have a well-defined notion of ‘a queue fed by Brownian motion’

(also called ‘reflected Brownian motion’).

Queue with stationary increments: $X_t - X_{t-s}$ has, for a given s, the same distribution, irrespective of t.

Steady-state distribution $Q \equiv \lim_{t\to\infty} Q_t$ exists if $\mathbb{E}X_1 < 0$.

$(Q_t)_t$ often referred to as *workload process.*

DEPENDENCE STRUCTURE

What happens if we feed the queue by a highly correlated input process $(X_t)_t$?

DEPENDENCE STRUCTURE

What happens if we feed the queue by a highly correlated input process $(X_t)_t$?

Is this high correlation inherited by the workload process?

DEPENDENCE STRUCTURE

Example: fractional Brownian motion. Then $X_t \sim \mathrm{Norm}(-t, |t|^{2H})$, where $H \in (0,1)$ is the so-called *Hurst parameter.*

DEPENDENCE STRUCTURE

Example: fractional Brownian motion. Then $X_t \sim \mathrm{Norm}(-t, |t|^{2H})$, where $H \in (0,1)$ is the so-called *Hurst parameter.*

$$c_t := \mathbb{C}\mathrm{ov}\left(X_{t+\varepsilon} - X_t, X_\varepsilon - X_0\right) \sim t^{2H-2}.$$

In particular, c_t is nonsummable for $H \in (\frac{1}{2}, 1)$. We say: (the rate process of) $(X_t)_t$ is *long-range dependent.*

DEPENDENCE STRUCTURE

Example: fractional Brownian motion. Then $X_t \sim \mathrm{Norm}(-t, |t|^{2H})$, where $H \in (0,1)$ is the so-called *Hurst parameter.*

$$c_t := \mathbb{C}\mathrm{ov}\left(X_{t+\varepsilon} - X_t, X_\varepsilon - X_0\right) \sim t^{2H-2}.$$

In particular, c_t is nonsummable for $H \in (\frac{1}{2}, 1)$. We say: (the rate process of) $(X_t)_t$ is *long-range dependent.*

Key question: what about

$$r(t) := \mathbb{C}\mathrm{ov}\left(Q_0, Q_t\right)??$$

Is the workload process long-range dependent?

DEPENDENCE STRUCTURE

Key question: what about

$r(t) := \mathbb{C}\mathrm{ov}\,(Q_0, Q_t)??$

Is the workload process long-range dependent?

Conjecture (M., Norros, & Glynn, *Ann. Appl. Prob. '09*):

$r(t) \sim t^{2H-2};$

that is, workload process is long-range dependent for $H \in (\frac{1}{2}, 1)$.

DEPENDENCE STRUCTURE

Was not proven yet.

Main difficulty: not straightforward to write $\mathbb{C}\mathrm{ov}\,(Q_0, Q_t)$ as large-deviation probability.

Therefore: other techniques are needed.

DEPENDENCE STRUCTURE

Was not proven yet.

Main difficulty: not straightforward to write $\mathbb{C}\mathrm{ov}\,(Q_0, Q_t)$ as large-deviation probability.

Therefore: other techniques are needed.

There are partial results, though, see thesis Es-Saghouani. Focus on probabilities of the type

$$\mathbb{P}(Q_0 > pB, Q_{TB} > qB)$$

for B large. Assume queue is in stationarity at time 0!

DEPENDENCE STRUCTURE

There are partial results, though, see thesis Es-Saghouani. Focus on probabilities of the type

$$\mathbb{P}(Q_0 > pB, Q_{TB} > qB)$$

for B large. Assume queue is in stationarity at time 0!

There large-deviations theory *is* applicable.

$$\mathbb{P}(Q_0 > pB, Q_{TB} > qB) - \mathbb{P}(Q > pB)\mathbb{P}(Q > qB)$$

is some sort of covariance for indicator functions.

THIS TALK

Same question, but now the queue is driven by a Lévy process (stationary and *independent* increments).

THIS TALK

Same question, but now the queue is driven by a Lévy process (stationary and *independent* increments).

Now input process is (evidently) *not* long-range dependent...

THIS TALK

Same question, but now the queue is driven by a Lévy process (stationary and *independent* increments).

Now input process is (evidently) *not* long-range dependent...

... but what can be said about workload process?

SPECTRALLY ONE-SIDED LÉVY PROCESSES

Lévy process: stochastic process with independent and identically distributed increments.

SPECTRALLY ONE-SIDED LÉVY PROCESSES

Lévy process: stochastic process with independent and identically distributed increments.

$(X_t)_t$: Lévy process without one-sided jumps; drift $\mathbb{E}X_1 < 0$.

Key object in applied probability.

SPECTRALLY ONE-SIDED LÉVY PROCESSES

Lévy process: stochastic process with independent and identically distributed increments.

$(X_t)_t$: Lévy process without one-sided jumps; drift $\mathbb{E}X_1 < 0$.

Key object in applied probability.

Spectrally one-sided: jumps are *either* only positive (spectrally positive)

or only negative (spectrally negative).

SPECTRALLY ONE-SIDED LÉVY PROCESSES: EXAMPLES

Important examples of spectrally one-sided Lévy processes are:

- ⋆ *Brownian motion with drift* (being actually both spectrally positive and negative).
- ⋆ *Compound Poisson with drift*. Non-negative jobs arrive according to a Poisson process of rate λ; the jobs $B_1, B_2, \ldots$ are i.i.d. samples from a distribution with Laplace transform $b(\alpha) := \mathbb{E}e^{-\alpha B}$; the storage system is continuously depleted at a rate 1.

  If drift would be positive, and jobs would be i.i.d. samples from a non-positive distribution (i.e., the jumps are downward), the process is spectrally negative.

INDUCED QUEUEING PROCESS

$(Q_t)_t$ denotes the *reflection* of spectrally one-sided Lévy process $(X_t)_t$ at 0:

$$Q_t := X_t + \max\{Q_0, -M_t\},\ t \geq 0,$$

where $M = (M_t)_{t\geq 0}$ is the decreasing process defined by $M_t := \inf_{0\leq u\leq t} X_u$.

We assume that the queue is already in stationarity at time $t = 0$.

SPECTRALLY ONE-SIDED

Spectrally-positive case:

Laplace exponent: φ. Thus $\varphi(\alpha) := \log \mathbb{E}e^{-\alpha X_1}$.
Increasing and convex on $[0, \infty)$, with slope $-\mathbb{E}X_1$ in the origin.
Inverse is ψ.

Assume that X_t is not a subordinator, i.e., a monotone process.

SPECTRALLY ONE-SIDED

Spectrally-positive case:

Laplace exponent: φ. Thus $\varphi(\alpha) := \log \mathbb{E}e^{-\alpha X_1}$.
Increasing and convex on $[0, \infty)$, with slope $-\mathbb{E}X_1$ in the origin.
Inverse is ψ.

Stationary behavior is well understood;

the steady-state distribution of $Q := \lim_{t\to\infty} Q_t$ is given through

$$\kappa(s) := \mathbb{E}e^{-sQ} = s\frac{\varphi'(0)}{\varphi(s)},\ s \geq 0.$$

Generalized Pollaczek-Khinchine formula.

SPECTRALLY ONE-SIDED

Spectrally-positive case, ctd.:

The steady-state distribution of $Q := \lim_{t\to\infty} Q_t$ is given through

$$\kappa(s) := \mathbb{E}e^{-sQ} = s\frac{\varphi'(0)}{\varphi(s)},\ s \geq 0.$$

Generalized Pollaczek-Khinchine formula.

Mean and variance:

$$\mu := \mathbb{E}Q = \frac{\varphi''(0)}{2\varphi'(0)};\ v := \mathbb{V}\mathrm{ar}Q = \left(\frac{\varphi''(0)}{2\varphi'(0)}\right)^2 - \frac{\varphi'''(0)}{3\varphi'(0)}.$$

We assume that $v < \infty$, i.e., $\varphi'''(0) < \infty$.

Brownian motion: $\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha^2\sigma^2$;

Compound Poisson: $\varphi(\alpha) = \alpha - \lambda + \lambda b(\alpha)$.

SPECTRALLY ONE-SIDED

Spectrally-negative case:

Define $\Phi(\beta) := \log \mathbb{E} e^{\beta X_1}$, for $\beta \geq 0$.

Again rule out that X_t is a subordinator (and recalling that $\Phi'(0) = \mathbb{E}X_1 < 0$).

Note: $\Phi(\beta)$ is no bijection on $[0, \infty)$. Therefore define the *right* inverse through

$$\Psi(q) := \sup\{\beta \geq 0 : \Phi(\beta) = q).$$

Realize that $\beta_0 := \Psi(0) > 0$.

SPECTRALLY ONE-SIDED

Spectrally-negative case, ctd.:

$\mathbb{E}e^{\beta_0 X_t}$ is martingale, with $\beta_0 := \Psi(0) > 0$.

'Optional sampling' thus gives, for any positive x, $\mathbb{P}(\exists t \geq 0 : X_t > x)e^{\beta_0 x} = 1$.

SPECTRALLY ONE-SIDED

Spectrally-negative case, ctd.:

$\mathbb{E}e^{\beta_0 X_t}$ is martingale, with $\beta_0 := \Psi(0) > 0$.

'Optional sampling' thus gives, for any positive x, $\mathbb{P}(\exists t \geq 0 : X_t > x)e^{\beta_0 x} = 1$.

Now Q is distributed as the supremum over $t \geq 0$ of X_t ('Reich's identity').

Hence: Q is exponentially distributed with mean $1/\beta_0$.

It follows that $v = 1/\beta_0^2$.

CORRELATION FUNCTION

This rest of this talk focuses on

- $\star$ structural properties of the workload covariance function;
- $\star$ efficient computation of the workload covariance function, through simulation.

CORRELATION FUNCTION

This rest of this talk focuses on

- ⋆ structural properties of the workload covariance function;
- ⋆ efficient computation of the workload covariance function, through simulation.

Recall: correlation function is

$$r(t) := \frac{\mathbb{C}\mathrm{ov}(Q_0, Q_t)}{\sqrt{\mathbb{V}\mathrm{ar}Q_0 \mathbb{V}\mathrm{ar}Q_t}} = \frac{\mathbb{E}(Q_0 Q_t) - (\mathbb{E}Q_0)^2}{\mathbb{V}\mathrm{ar}Q_0}.$$

Its Laplace transform is

$$\rho(\vartheta) = \int_0^\infty e^{-\vartheta t} r(t) \mathrm{d}t.$$

CORRELATION FUNCTION

Theorem: In the spectrally-positive case, the Laplace transform $\rho(\vartheta)$ of $r(t)$ is given by

$$\rho(\vartheta) = \frac{1}{\vartheta} - \frac{\varphi''(0)}{2v\vartheta^2} + \frac{\varphi'(0)}{v\vartheta^2}\left(\frac{1}{\vartheta\psi'(\vartheta)} - \frac{1}{\psi(\vartheta)}\right).$$

(See Es Saghouani and M., *J. Appl. Prob.*, 2008).

CORRELATION FUNCTION, ctd.

- For $T \sim \text{Exp}(\vartheta)$ independent of X_t we have (Kella, Boxma, and M., *J. Appl. Prob.*, 2006)

$$\mathbb{E}(e^{-sQ_T}|Q_0 = q) = \frac{\vartheta}{\vartheta - \varphi(s)} \left(e^{-sq} - s\frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)} \right). \tag{1}$$

- Differentiate (1) w.r.t. s and let $s \downarrow 0$ yields

$$\int_0^\infty \vartheta e^{-\vartheta t}\mathbb{E}(Q_t \mid Q_0 = q)\mathrm{d}t = -\frac{\varphi'(0)}{\vartheta} + q + \frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)}.$$

- The result follows from

$$\int_0^\infty e^{-\vartheta t}\mathbb{E}(Q_0Q_t)\mathrm{d}t = \int_0^\infty \frac{q}{\vartheta}\int_0^\infty \vartheta e^{-\vartheta t}\mathbb{E}(Q_t \mid Q_0 = q)\mathrm{d}t\mathrm{d}\mathbb{P}(Q_0 \leq q).$$

CORRELATION FUNCTION, ctd.

Example: Brownian motion

$$\rho(\vartheta) = \frac{1}{\vartheta} - \frac{2}{\vartheta^2} + \frac{2}{\vartheta^3}\left(\sqrt{1+2\vartheta} - 1\right)$$

which we can explicitly invert to obtain

$$r(t) = 2(1-2t-t^2)\left(1 - \Phi_N(\sqrt{t})\right) + 2\sqrt{t}(1+t)\phi_N(\sqrt{t}),$$

where $\Phi_N(\cdot)$ (resp. $\phi_N(\cdot)$) is the standard Normal distribution (resp. density).

CORRELATION FUNCTION, ctd.

Es-Saghouani and M.: structural properties for spectrally-positive case, such as $r(t)$ is positive, decreasing, convex, complementing M/G/1 results by Ott.

Relying on machinery of completely monotone functions (Bernstein, 1929). (We'll demonstrate this concept for the spectrally-negative case.)

In addition: asymptotics of $r(t)$ for t large.

CORRELATION FUNCTION

But how about spectrally-negative case?

CORRELATION FUNCTION

But how about spectrally-negative case?

q-scale functions: $W^{(q)}(x)$ is a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\Phi(\beta) - q}, \qquad \beta > \Psi(q), \tag{2}$$

and in addition

$$Z^{(q)}(x) := 1 + q + \int_0^x W^{(q)}(y)\mathrm{d}y. \tag{3}$$

CORRELATION FUNCTION: TRANSFORM

Pistorius, *J. Th. Prob.*, 2004: transform (with respect to t) of the density of Q_t, given that $Q_0 = x$:

$$\int_0^\infty e^{-qt}\mathbb{P}_x(Q_t = y)\mathrm{d}t = e^{-\Psi(q)y}\frac{\Psi(q)}{q}Z^{(q)}(x) - W^{(q)}(x-y).$$

CORRELATION FUNCTION: TRANSFORM

Pistorius, *J. Th. Prob.*, 2004: transform (with respect to t) of the density of Q_t, given that $Q_0 = x$:

$$\int_0^\infty e^{-qt}\mathbb{P}_x(Q_t = y)\mathrm{d}t = e^{-\Psi(q)y}\frac{\Psi(q)}{q}Z^{(q)}(x) - W^{(q)}(x-y).$$

Hence, with T an exponential random variable with mean q^{-1},

$$\int_0^\infty e^{-\beta x}\mathbb{E}_x e^{-\alpha Q_T}\mathrm{d}x = I_1 - I_2;$$

here

$$I_1 := \int_0^\infty \int_0^\infty q e^{-\beta x}e^{-\alpha y}e^{-\Psi(q)y}\frac{\Psi(q)}{q}Z^{(q)}(x)\mathrm{d}x\mathrm{d}y = ... = \frac{\Psi(q)}{\Psi(q)+\alpha}\frac{1}{\beta}\left(1 + \frac{q}{\Phi(\beta)-q}\right),$$

$$I_2 := \int_0^\infty \int_0^\infty q e^{-\beta x}e^{-\alpha y}W^{(q)}(x)\mathrm{d}x\mathrm{d}y = ... = \frac{q}{\alpha+\beta}\frac{1}{\Phi(\beta)-q}.$$

CORRELATION FUNCTION: TRANSFORM

So we have an expression for

$$\int_0^\infty \beta e^{-\beta x} \mathbb{E}_x e^{-\alpha Q_T} \mathrm{d}x.$$

CORRELATION FUNCTION: TRANSFORM

So we have an expression for

$$\int_0^\infty \beta e^{-\beta x} \mathbb{E}_x e^{-\alpha Q_T} \mathrm{d}x.$$

Sanity checks:

- Plugging in $\alpha = 0$ yields 1.
- Plugging in $\beta = \beta_0$ yields the steady-state transform $\beta_0/(\beta_0 + \alpha)$:

  when starting in the queue's equilibrium distribution at time 0, the workload is still in stationarity after an exponentially distributed time (irrespectively of q).

CORRELATION FUNCTION: TRANSFORM

With T having an exponential distribution with mean q^{-1},

$$\int_0^\infty qe^{-qt}\mathbb{E}(Q_0Q_t)\mathrm{d}t = \int_0^\infty \beta_0 x e^{-\beta_0 x}\mathbb{E}_x Q_T\,\mathrm{d}x = \lim_{\alpha\downarrow 0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\beta\cdot\frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty e^{-\beta x}\mathbb{E}_x e^{-\alpha Q_T}\mathrm{d}x\bigg|_{\beta=\beta_0}\right].$$

CORRELATION FUNCTION: TRANSFORM

With T having an exponential distribution with mean q^{-1},

$$\int_0^\infty q e^{-qt}\mathbb{E}(Q_0 Q_t)\mathrm{d}t = \int_0^\infty \beta_0 x e^{-\beta_0 x}\mathbb{E}_x Q_T\,\mathrm{d}x = \lim_{\alpha\downarrow 0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\beta\cdot\frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty e^{-\beta x}\mathbb{E}_x e^{-\alpha Q_T}\mathrm{d}x\bigg|_{\beta=\beta_0}\right].$$

We eventually find, after considerable calculus, the following result.

Theorem: In the spectrally-negative case, the Laplace transform $\rho(\vartheta)$ of $r(t)$ is given by

$$\rho(q) := \int_0^\infty r(t)\,e^{-qt}\mathrm{d}t = \frac{1}{q} + \frac{\beta_0^2}{q^2}\Phi'(\beta_0)\left(\frac{1}{\Psi(q)} - \frac{1}{\beta_0}\right).$$

CORRELATION FUNCTION: TRANSFORM

Corollary: For the spectrally-negative case,

$$\rho(0) := \int_0^\infty r(t)\mathrm{d}t = \frac{1}{\beta_0 \Phi'(\beta_0)} + \frac{\Phi''(\beta_0)}{2(\Phi'(\beta_0))^3} < \infty.$$

CORRELATION FUNCTION: STRUCTURAL PROPERTIES

Theorem: $r(\cdot)$ is positive, decreasing, and convex.

Proof: Mimic the proof in Es-Saghouani and M. for the spectrally-positive case. $\mathscr{C}$: class of completely monotone functions.

CORRELATION FUNCTION: STRUCTURAL PROPERTIES

Integration by parts:

$$\rho^{(1)}(q) := \int_0^\infty r'(t)\mathrm{d}t = \frac{\beta_0^2}{q}\Phi'(\beta_0)\left(\frac{1}{\Psi(q)} - \frac{1}{\beta_0}\right);$$

$$\rho^{(2)}(q) := \int_0^\infty r''(t)\mathrm{d}t = -r'(0) + \beta_0^2\Phi'(\beta_0)\left(\frac{1}{\Psi(q)} - \frac{1}{\beta_0}\right).$$

Later we show that $\Psi(0)/\Psi(q) \in \mathscr{C}$.

Conclude: $\rho^{(2)}(q)$ is in $\mathscr{C}$, and hence $r''(\cdot)$ is positive, i.e., $r(\cdot)$ is convex.

Known: $f(q) \in \mathscr{C}$ implies that, with $g(q) := (f(0) - f(q))/q$, also $g(q) \in \mathscr{C}$.

Taking $f(q) = \rho^{(2)}(q)$, we have $-\rho^{(1)}(q)$ is in $\mathscr{C}$, and hence $r'(\cdot)$ is negative, i.e., $r(\cdot)$ is decreasing.

Similarly, $\rho(q)$ is in $\mathscr{C}$, and hence $r(\cdot)$ is positive. $\square$

BUSY PERIOD: AN INTERMEZZO

$\tau := \inf\{t \geq 0 : Q_t = 0\}$, where Q_0 has stationary distribution.

$p(t) := \mathbb{P}(\tau > t)$.

BUSY PERIOD: AN INTERMEZZO

$\tau := \inf\{t \geq 0 : Q_t = 0\}$, where Q_0 has stationary distribution.

$p(t) := \mathbb{P}(\tau > t)$.

Spectrally positive:

$$\int_0^\infty e^{-\vartheta t} p(t)\mathrm{d}t = \int_0^\infty \left(\int_0^\infty e^{-\vartheta t} \mathbb{P}(\tau(x) > t)\mathrm{d}t \right) \mathrm{d}\mathbb{P}(Q_0 < x) = \frac{1}{\vartheta} \int_0^\infty \left(1 - e^{-\psi(\vartheta) q} \right) \mathrm{d}\mathbb{P}(Q_0 < q).$$

Application of 'Pollaczek-Khinchine':

$$\int_0^\infty e^{-\vartheta t} p(t)\mathrm{d}t = \frac{1}{\vartheta} - \varphi'(0) \frac{\psi(\vartheta)}{\vartheta^2}.$$

BUSY PERIOD: AN INTERMEZZO

Spectrally negative:

Recall that $\int_0^\infty e^{-qt}\mathbb{P}(\tau > t)\mathrm{d}t = q^{-1}\left(1 - \mathbb{E}e^{-q\tau}\right)$.

Known result:

$$\mathbb{E}e^{-q\tau} = \int_0^\infty \beta_0 e^{-\beta_0 x}\mathbb{E}e^{-q\tau(x)}\mathrm{d}x = \beta_0 \cdot \frac{\hat{\kappa}(q,\beta_0) - \hat{\kappa}(q,0)}{\beta_0\hat{\kappa}(q,\beta_0)};$$

here $\hat{\kappa}(q,\beta)$ relates to the transform of the so-called *descending ladder process*;

in spectrally-negative case: $\hat{\kappa}(q,\beta) = (q - \Phi(\beta))/(\Psi(q) - \beta)$. Using that $\Phi(\beta_0) = 0$, we find

$$\mathbb{E}e^{-q\tau} = \Psi(0)/\Psi(q),$$

and in addition

$$\int_0^\infty e^{-qt}p(t)\mathrm{d}t = \frac{1}{q}\left(1 - \frac{\Psi(0)}{\Psi(q)}\right).$$

BUSY PERIOD: AN INTERMEZZO

Striking feature: transforms have the same *branching point* as the transforms of the correlation function!!

BUSY PERIOD: AN INTERMEZZO

Striking feature: transforms have the same *branching point* as the transforms of the correlation function!!

Spectrally positive, light tails ($\exists \alpha < 0 : \varphi(\alpha) = 0$): we roughly have

$$r(t) \sim p(t) \sim e^{\vartheta^\star t},$$

where ζ is the minimizer of $\varphi(\cdot)$ and $\vartheta^\star = \varphi(\zeta)$ the branching point of $\psi(\cdot)$.

BUSY PERIOD: AN INTERMEZZO

Striking feature: transforms have the same *branching point* as the transforms of the correlation function!!

Spectrally positive, light tails ($\exists \alpha < 0 : \varphi(\alpha) = 0$): we roughly have

$$r(t) \sim p(t) \sim e^{\vartheta^\star t},$$

where ζ is the minimizer of $\varphi(\cdot)$ and $\vartheta^\star = \varphi(\zeta)$ the branching point of $\psi(\cdot)$.

Spectrally negative: we roughly have

$$r(t) \sim p(t) \sim e^{q^\star t},$$

where ζ is the minimizer of $\Phi(\cdot)$ and $q^\star = \Phi(\zeta)$ the branching point of $\Psi(\cdot)$.

BUSY PERIOD: AN INTERMEZZO

Naive simulation: estimate $p(t)$ by

$$S_n^{(\mathrm{NS})}(t) := \frac{1}{n} \sum_{i=1}^{n} 1\{\tau_i > t\}.$$

BUSY PERIOD: AN INTERMEZZO

Naive simulation: estimate $p(t)$ by

$$S_n^{(\mathrm{NS})}(t) := \frac{1}{n}\sum_{i=1}^{n} 1\{\tau_i > t\}.$$

Number of runs needed to obtain estimate of given precision?

$$\frac{\sqrt{\mathbb{V}\mathrm{ar} S_n^{(\mathrm{NS})}(t)}}{p(t)} < \varepsilon,$$

and realizing that

$$\mathbb{V}\mathrm{ar} S_n^{(\mathrm{NS})}(t) = \frac{1}{n} p(t)(1 - p(t)) \approx \frac{p(t)}{n},$$

we see that the number of runs needed is roughly of order $1/p(t)$, i.e., exponentially increasing...

BUSY PERIOD: AN INTERMEZZO

A more clever algorithm can be constructed as follows (spectrally-positive case):

use Importance Sampling.

- $\star$ Let, in the interval $(0,t]$, the Lévy process be twisted with $-\zeta = -\psi(\vartheta^\star) > 0$.
  Meaning: $\varphi(\vartheta)$ replaced by $\bar{\varphi}(\vartheta) := \varphi(\vartheta+\zeta) - \varphi(\zeta)$.
- $\star$ But what about distribution of Q_0?
  Simulate Q_0 from a κ-twisted version, i.e., a distribution with LT $\mathbb{E}e^{-(\alpha-\kappa)Q_0}/\mathbb{E}e^{\kappa Q_0}$.

Call new measure $\mathbb{Q}_\kappa$.

BUSY PERIOD: AN INTERMEZZO

We simulate the process under $\mathbb{Q}_\kappa$ till time t. Likelihood $L := L_A \cdot L_B$, where

- ⋆ contribution due to the twisted Lévy process between 0 and t:

$$L_A := e^{\psi(\vartheta^\star)X_t} \cdot \mathbb{E}e^{-\psi(\vartheta^\star)X_t} = e^{\psi(\vartheta^\star)X_t} \cdot e^{\vartheta^\star t}.$$

- ⋆ contribution due to the twisted queue at time 0 (use 'Pollaczek-Khinchine'):

$$L_B := e^{-\kappa Q_0} \cdot \mathbb{E}e^{\kappa Q_0} = e^{-\kappa Q_0} \cdot \frac{-\kappa\varphi'(0)}{\varphi(-\kappa)}.$$

Estimate $p(t)$ by, sampling under $\mathbb{Q}_\kappa$,

$$S_n^{(\mathrm{IS})}(t) := \frac{1}{n}\sum_{i=1}^{n} L_i 1\{\tau_i > t\}.$$

BUSY PERIOD: AN INTERMEZZO

$$L = e^{\psi(\vartheta^\star)X_t} \cdot e^{\vartheta^\star t} \cdot e^{-\kappa Q_0} \cdot \frac{-\kappa\varphi'(0)}{\varphi(-\kappa)}.$$

First option: not twisting Q_0 at all (i.e., choosing $\kappa = 0$).

This does *not* work well: recalling that a necessary condition for $\{\tau > t\}$ is $\{Q_0 + X_t > 0\}$, we find

$$\mathbb{E}_{\mathbb{Q}_\kappa} L^2 1\{\tau > t\} \leq \left(-\frac{\kappa\varphi'(0)}{\varphi(-\kappa)}\right)^2 e^{2\vartheta^\star t}\mathbb{E}_{\mathbb{Q}_\kappa} e^{-2\kappa Q_0} e^{-2\psi(\vartheta^\star)Q_0}. \tag{4}$$

Logarithmic efficiency, meaning that the number of replications needed to obtain an estimate with a certain fixed precision grows subexponentially in the 'rarity parameter' t:

$$\limsup_{t\to\infty} t^{-1} \log \mathbb{E}_{\mathbb{Q}_\kappa} L^2 1\{\tau > t\} \leq 2\vartheta^\star.$$

In other words: when picking $\kappa = 0$ we need to have $\mathbb{E}_{\mathbb{Q}_0} e^{-2\psi(\vartheta^\star)Q_0} < \infty$ for logarithmic efficiency... *Not* a priori clear....

BUSY PERIOD: AN INTERMEZZO

$$L = e^{\psi(\vartheta^\star)X_t} \cdot e^{\vartheta^\star t} \cdot e^{-\kappa Q_0} \cdot \frac{-\kappa\varphi'(0)}{\varphi(-\kappa)}.$$

Second option: twisting with $\kappa = -\zeta > 0$.

Easy to see that we do get logarithmic efficiency here!

SIMULATION OF CORRELATION FUNCTION

But can we come up with an efficient simulation algorithm for $r(t)$?

Remember:

$$r(t) = \frac{\mathbb{E}Q_0Q_t - \mu^2}{v},$$

with $\mu := \mathbb{E}Q$ and $v := \mathbb{V}\mathrm{ar}Q$ known...

We can estimate $\mathbb{E}Q_0Q_t - \mu^2$ by

$$T_n^{(\mathrm{NS})}(x) := \frac{1}{n}\sum_{i=1}^{n} Q_0^{(i)}Q_t^{(i)} - \mu^2.$$

How many runs needed?

SIMULATION OF CORRELATION FUNCTION

Variance of this estimator:

$$\frac{1}{n} \cdot \mathbb{V}\mathrm{ar}\,(Q_0 Q_t) = \frac{\mathbb{E}(Q_0^2 Q_t^2) - (\mathbb{E}(Q_0 Q_t))^2}{n} \to \frac{(\mathbb{E}Q^2)^2 - (\mathbb{E}Q)^4}{n};$$

Conclude: number of runs needed roughly proportional to $1/r(t)^2$!!!

SIMULATION OF CORRELATION FUNCTION

We construct a coupling as follows.

Write:

$$r(t) = \frac{1}{v} \cdot \mathbb{E}(Q_0 \cdot (Q_t - Q_t^\star)),$$

where both Q and $Q^\star$ are stationary versions of the workload, and $Q_t^\star$ is *independent* of Q_0.

Construct this as follows: generate Q_0 and $Q_0^\star$ independently, sampled from the stationary distribution of the workload. Now use exactly the same driving Lévy process X_t over $(0, t]$ to drive both Q_t and $Q_t^\star$ from their two independently generated initial conditions.

This makes Q_t and Q_0 correlated but $Q_t^\star$ and Q_0 independent.

SIMULATION OF CORRELATION FUNCTION

We can estimate $\mathbb{E}Q_0 Q_t - \mu^2$ by

$$T_n^{(\mathrm{CS})}(x) := \frac{1}{n}\sum_{i=1}^{n} Q_0^{(i)}(Q_t^{(i)} - Q_t^{\star(i)}).$$

What is performance of this estimator?

SIMULATION OF CORRELATION FUNCTION

Split $\mathbb{E}(Q_0 \cdot (Q_t - Q_t^\star))$ into four terms, as follows.

Recall $M_t = \inf_{s\in(0,t]} X_s$. Then

$$r(t) = r_{++}(t) + r_{+-}(t) + r_{-+}(t) + r_{--}(t),$$

where

$$\begin{aligned}
r_{++}(t) &:= \mathbb{E}(Q_0 \cdot (Q_t - Q_t^\star) \cdot 1\{Q_0 + M_t > 0, Q_0^\star + M_t > 0\}), \\
r_{+-}(t) &:= \mathbb{E}(Q_0 \cdot (Q_t - Q_t^\star) \cdot 1\{Q_0 + M_t > 0, Q_0^\star + M_t < 0\}), \\
r_{-+}(t) &:= \mathbb{E}(Q_0 \cdot (Q_t - Q_t^\star) \cdot 1\{Q_0 + M_t < 0, Q_0^\star + M_t > 0\}), \\
r_{--}(t) &:= \mathbb{E}(Q_0 \cdot (Q_t - Q_t^\star) \cdot 1\{Q_0 + M_t < 0, Q_0^\star + M_t < 0\}).
\end{aligned}$$

It is evident that $r_{--}(t) = 0$ as both queues have been empty.

SIMULATION OF CORRELATION FUNCTION

Key observation: $|Q_t - Q_t^\star| \leq |Q_0 - Q_0^\star|$.

We therefore have:

$$\mathbb{V}\mathrm{ar}\,(Q_0(Q_t - Q_t^\star) \leq \mathbb{E}Q_0^2(Q_t - Q_t^\star)^2 \leq \mathbb{E}Q_0^2(Q_0 - Q_0^\star)^2.$$

In addition:

$$\begin{aligned}\mathbb{E}Q_0^2(Q_0 - Q_0^\star)^2 \;\leq\; & \mathbb{E}(Q_0^2(Q_0 - Q_0^\star)^2 \cdot 1\{Q_0 + M_t > 0, Q_0^\star + M_t > 0\}) + \\ & + \mathbb{E}(Q_0^2(Q_0 - Q_0^\star)^2 \cdot 1\{Q_0 + M_t > 0, Q_0^\star + M_t \leq 0\}) \\ & + \mathbb{E}(Q_0^2(Q_0 - Q_0^\star)^2 \cdot 1\{Q_0 + M_t \leq 0, Q_0^\star + M_t > 0\})\end{aligned}$$

SIMULATION OF CORRELATION FUNCTION

Lemma: in the spectrally-positive case

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(Q_0^k 1\{\tau > t\}) \leq \vartheta^\star$$

(and $\ldots \leq q^\star$ in the spectrally-negative case).

Hence,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{V}\mathrm{ar}\,(Q_0(Q_t - Q_t^\star) \leq \vartheta^\star.$$

Consequently,

$$\frac{\sqrt{\mathbb{V}\mathrm{ar}\, T_n^{(\mathrm{CS})}(x)}}{r(t)} \approx \frac{\sqrt{e^{\vartheta^\star t}/n}}{e^{\vartheta^\star t}},$$

so that number of runs needed grows roughly as $1/r(t)$.

Substantial improvement!

SIMULATION OF CORRELATION FUNCTION

Augment coupling algorithm with Importance Sampling (as for busy period),

and we even get an algorithm for which the number of runs grows *subexponentially*!

This algorithm is called *logarithmically efficient.*

CONCLUSIONS

Analysis of correlation structure of process after imposing reflection map: is correlation structure inherited?

Hard to solve for the case of Gaussian inputs (such as fractional Brownian motion), but . . .

explicit analysis for the Lévy case: structural results, asymptotics, and efficient simulaton.