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CONTINUOUS-TIME MONOTONE STOCHASTIC RECURSIONS AND DUALITY

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Abstract

A duality is presented for continuous-time, real-valued, monotone, stochastic recursions driven by processes with stationary increments. A given recursion defines the time evolution of a content process (such as a dam or queue), and it is shown that the existence of the content process implies the existence of a corresponding dual risk process that satisfies a dual recursion. The one-point probabilities for the content process are then shown to be related to the one-point probabilities of the risk process. In particular, it is shown that the steady-state probabilities for the content process are equivalent to the first passage time probabilities for the risk process. A number of applications are presented that flesh out the general theory. Examples include regulated processes with one or two barriers, storage models with general release rate, and jump and diffusion processes.

Keywords: Loynes' lemma; jump-diffusion process; reflected process; risk process; ruin probability; Siegmund duality; stationary distribution; stationary increments; storage process; two-barrier reflection

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1. Introduction

Given a general measurable space $\boldsymbol{U}$ and a measurable function $f : [0, \infty) \times \boldsymbol{U} \longrightarrow [0, \infty)$, a stochastic *sequence* can be explicitly constructed recursively by

$$V_{n+1} = f(V_n, U_n), \quad n \geq 0, \tag{1.1}$$

where $\{U_n : n \in \mathbb{Z}\}$ is a given sequence (the *driving sequence*) of random elements taking values in $\boldsymbol{U}$.

Making sense of recursion in continuous time is more difficult for various reasons. First, any given f does not always yield a process recursively as in (1.1). Second, for a given function r and a given process $A = \{A_t : t \geq 0\}$, an equation such as

$$V_t = V_0 + A_t - \int_0^t r(V_s)\,\mathrm{d}s,$$

while being implicitly recursive, does not offer an explicit f. In fact, for certain choices of A and r establishing that a unique V exists may be difficult if not impossible.

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In this paper we study continuous-time, real-valued stochastic processes $\{V_t : t \geq 0\}$ that are *assumed* to be defined recursively from a non-negative monotone (in y) function $f(y, t, Z)$, where $y \geq 0$, $t \geq 0$ and $Z = \{Z_t : t \geq 0\}$ is a stochastic process with stationary increments, taking values in a measurable space. Our purpose here is to construct a dual recursive function g yielding a dual *risk* process $\{R_t : t \geq 0\}$, having (among other features) the property that

$$\mathrm{P}(V > x) = \mathrm{P}(\tau(x) < \infty),$$

where $\tau(x)$ denotes the time of *ruin* for the risk process starting initially with reserve x and V denotes an r.v. with the steady-state distribution (as $t \to \infty$) of V_t. Although more subtle, this work is analogous to that done by Asmussen and Sigman [8] in discrete time and, in the Markovian case, is related to duality in stochastically monotone Markov processes as studied by Siegmund [22]. The need for a continuous-time analog of [8] is discussed in [3], which is a nice general survey of duality with many references.

In Section 2 the recursion framework for the content process, together with the stationary construction, is given (we do not view this section as profound or really new). The dual recursion and duality are in Section 3 (with main results Proposition 3.1 and Corollary 3.1). Examples are given in Section 4, including a regulated process with one or two barriers, a storage model with general release rate, and jump and diffusion processes.

2. Continuous-time monotone recursions

Let $\boldsymbol{U}$ be a measurable space on which there is a notion of addition and subtraction, and let 0 denote the zero element of $\boldsymbol{U}$. Let

$$\boldsymbol{D}_0 = \{z : [0, \infty) \to \boldsymbol{U} : z_0 = 0\}$$

denote the space of functions with values in $\boldsymbol{U}$ that satisfy $z_0 = 0$. (In practice we shall often restrict the space $\boldsymbol{D}_0$ further, i.e. by defining some norm on $\boldsymbol{D}_0$ and requiring some sort of continuity on the paths of z. But these restrictions are unnecessary for the abstract construction of V_t and its dual R_t.) Next let

$$f : [0, \infty) \times [0, \infty) \times \boldsymbol{D}_0 \longrightarrow [0, \infty)$$

be a measurable function (denoted by $f(y, t, z)$), satisfying the following three conditions:

(A1) $f \geq 0$ and $f(y, 0, z) = y$.

(A2) $f(\cdot, t, z)$ is non-decreasing and left-continuous for each fixed $t \geq 0$, $z \in \boldsymbol{D}_0$.

(A3) (*Recursion*) For any $t \geq 0$, $h \geq 0$, $y \geq 0$ and $z \in \boldsymbol{D}_0$,

$$f(y, t + h, z) = f(f(y, t, z), h, \theta_t z),$$

where $\theta_t z = \{z_{t+s} - z_t : s \geq 0\}$ denotes the shifted (by t) increments of z.

While Conditions (A1) and (A2) are natural conditions, Condition (A3) is the all-important recursion assumption and does not follow (in general) from the first two conditions.

We have not yet mentioned probability measures or stochastic processes. All we have done is defined a recursive mapping from $\boldsymbol{D}_0$ into $(\mathbb{R}^+)^{[0,\infty)}$, the space of non-negative, real-valued functions. Given a stochastic process Z with paths in $\boldsymbol{D}_0$, i.e. given a triple $(\boldsymbol{D}_0, \mathcal{F}, \mathrm{P})$, (A3) allows us to define a stochastic process

$$V_t[y] \stackrel{\text{def}}{=} f(y, t, Z), \quad t \geq 0, \tag{2.1}$$

with paths in $(\mathbb{R}^+)^{[0,\infty)}$ and initial state $V_0[y] = y$. The probability measure P on $\boldsymbol{D}_0$ induces a probability measure on $(\mathbb{R}^+)^{[0,\infty)}$ via our mapping. This new process is recursive in the sense that

$$V_{t+h}[y] = f(V_t[y], h, \theta_t Z), \quad t \geq 0,\ h \geq 0. \tag{2.2}$$

Z is called the *driving* process for V_t. Note that we do not require any sort of continuity in t for $f(y, t, z)$ in order to construct V_t or its dual R_t. In practice, however, we shall always impose either left- and/or right-continuity on the paths of $f(y, \cdot, z)$.

Inherent in (A3) is the property that $f(y, t, z)$ only depends on z up to time t; in other words, $f(y, t, z)$ only depends on the increments $\{z_s - z_0 : 0 \leq s \leq t\}$ (recall that $z_0 = 0$). We assume that this is so, and for mathematical convenience we introduce the notation $\theta_{(a,b)}z$, $a < b$ to denote the increments between a and b:

$$\theta_{(a,b)}z_s \stackrel{\text{def}}{=} \begin{cases} z_{a+s} - z_a; & \text{if } 0 \leq s \leq b - a, \\ z_b - z_a; & \text{if } s > b - a. \end{cases} \tag{2.3}$$

Thus $\theta_{(a,b)}z$ is an element of $\boldsymbol{D}_0$ that is constant after time $b - a$, and

$$f(y, t, z) = f(y, t, \theta_{(0,t)}z)$$

and

$$V_{t+h}[y] = f(V_t[y], h, \theta_{(t,t+h)}Z). \tag{2.4}$$

Remarks.

1. Although we assume that f is defined for all $z \in \boldsymbol{D}_0$, it is sometimes necessary (in applications) to restrict z to a smaller subspace, such as those functions that are piecewise constant. These functions arise naturally in the context of queues, where z could be the counting process of a marked point process of arrivals. Such restrictions do not affect the general results in the present paper; the only requirement on the subspace of interest is that it be closed under the shift: for each $h \geq 0$, $\theta_h z = \{z_{h+s} - z_h : s \geq 0\}$ lies back in the subspace.

2. In the context of queueing models when the space $\boldsymbol{D}_0$ is the space of marked point processes on the real line and the z's are point processes, Condition (A3) can be found in [11, p. 143], and in [18] (where monotonicity is also assumed). Another general reference for recursions is [16].

Stationary construction of V

Assume Z (with paths in $\boldsymbol{D}_0$) is a stochastic process with stationary increments (this means that $\{Z_{s+t} - Z_s : t \geq 0\}$ has the same distribution as Z for all $s \geq 0$). Let $V_t[y] = f(y, t, Z)$, $t \geq 0$ be the content process of interest. As is standard, we now assume that Z has been extended to a two-sided process $\{Z_t : -\infty < t < \infty\}$ by the use of Kolmogorov's extension theorem. In this case the shifted increments $\theta_s Z = \{Z_{s+t} - Z_s : t \in \mathbb{R}\}$ are defined for all $s \in \mathbb{R}$. For $s \le t \in \mathbb{R}$, let

$$V_t^{(s)}[y] \stackrel{\text{def}}{=} f(y, t-s, \theta_s Z) = f(y, t-s, \theta_{(s,t)} Z). \tag{2.5}$$

$V_t^{(s)}[y]$ denotes the content level at time t if the content is initially y at time s, $-\infty < s < t$, and uses $\theta_s Z$ to drive its recursion up to time t. By stationarity of increments $V_t^{(s)}[y]$ has the same distribution as $V_{t-s}[y]$ for every fixed s and t, $s \le t$. In particular, when $t = 0$, we see that

$$V_0^{(-s)}[y] \text{ has the same distribution as } V_s[y] \text{ for every fixed } s \ge 0. \tag{2.6}$$

The case $y = 0$ is special:

Lemma 2.1. *For each $t \in \mathbb{R}$, $V_t^{(s)}[0]$ is non-decreasing as $s \to -\infty$.*

Proof. For $h > 0$, (A3) yields

$$V_t^{(s-h)}[0] = f(0, t-(s-h), \theta_{s-h} Z) = f(f(0, h, \theta_{s-h} Z), t-s, \theta_s Z).$$

By non-negativity of f (from (A1)) and monotonicity (from (A2))

$$f(f(0, h, \theta_{s-h} Z), t-s, \theta_s Z) \ge f(0, t-s, \theta_s Z) = V_t^s[0],$$

completing the proof.

Just as Loynes' lemma (Lemma 1 in [17], Section 6.2 in [23] and see [18] for a continuous-time queueing framework) did in discrete time, the above lemma allows us to construct a two-sided stationary version $\{V_t^* : t \in \mathbb{R}\}$ of V jointly with Z, via

$$V_t^* \stackrel{\text{def}}{=} \lim_{s \to -\infty} V_t^{(s)}[0], \tag{2.7}$$

such that the recursive property

$$V_{t+h}^* = f(V_t^*, h, \theta_t Z), \quad t \in \mathbb{R}, h \ge 0, \tag{2.8}$$

still holds. To see this, we note that

$$\lim_{s \to -\infty} V_{t+h}^{(s)}[0] = \lim_{s \to -\infty} f(V_t^{(s)}[0], h, \theta_t Z) = f\left(\lim_{s \to -\infty} V_t^{(s)}[0], h, \theta_t Z\right).$$

This follows from the fact that $f(y, t, Z)$ is left-continuous in y and $V_t^{(s)}[0]$ is non-decreasing as $s \to -\infty$.

Finally, we note that Equation (2.6) and Lemma 2.1 imply that $V_t[0]$ monotonically increases in distribution (as $t \to \infty$) to the distribution of V_0^*, thus confirming that V_0^* has the limiting distribution of $V_t[0]$. We let V denote a generic random variable with this limiting distribution and note that it is possible that $\mathrm{P}(V = \infty) > 0$. Stability conditions ensuring that $\mathrm{P}(V < \infty) = 1$ are model dependent, and we will deal with such issues only in the pure diffusion case (see Example 3 below).

3. The dual recursion

Given our recursive function $f(y, t, z)$ satisfying Conditions (A1)–(A3), we define for each $x \geq 0$ the dual function g as

$$g(x, t, z) = \sup\{y \geq 0 : f(y, t, \theta_{(-t,0)}z) \leq x\}, \tag{3.1}$$

where $\sup\{\emptyset\} = -\infty$.

The function $g(\cdot, t, z)$ is the right-continuous inverse function of $f(\cdot, t, \theta_{(-t,0)}z)$, and

$$g : ([0, \infty] \cup \{-\infty\}) \times [0, \infty) \times \boldsymbol{D}_0 \longrightarrow [0, \infty] \cup \{-\infty\},$$

where $g(-\infty, t, z) = -\infty$ and $g(+\infty, t, z) = +\infty$.

It is immediate that $g(x, 0, z) = x$, and the following can be proved exactly as is Lemma 2.1 in [8].

Lemma 3.1. *The following properties hold:*

1. $f(y, t, \theta_{(-t,0)}z) \leq x$ *if and only if* $g(x, t, z) \geq y$.
2. $g(x, t, z) = -\infty$ *if and only if* $f(0, t, \theta_{(-t,0)}z) > x$.
3. $g(x, 0, z) = x$, *and for each fixed* $t \geq 0$, $z \in \boldsymbol{D}_0$, $g(\cdot, t, z)$ *is a non-decreasing, right-continuous function.*

The next proposition is crucial in that it allows one to define all the finite-dimensional distributions of the dual process $R[x]$ (defined below) of $V[y]$.

Proposition 3.1. *The dual function g satisfies an inverse recursion relation, i.e. for any* $t \geq 0$, $h \geq 0$, $y \geq 0$ *and* $z \in \boldsymbol{D}_0$,

$$g(x, t + h, z) = g(g(x, t, z), h, \theta_{-t}z). \tag{3.2}$$

Proof. The case when $x = +\infty$ or $-\infty$ is immediate, so we assume that $x \in [0, \infty)$. Because f satisfies (A3) and $f(y, t, \theta_h\theta_{(-t-h,0)}z) = f(y, t, \theta_{(-t,0)}z)$, we have

$$\begin{aligned} g(x, t + h, z) &= \sup\{y \geq 0 : f(f(y, h, \theta_{(-t-h,0)}z), t, \theta_{(-t,0)}) \leq x\} \\ &= \sup\{y \geq 0 : f(y, h, \theta_{(-t-h,0)}z) \leq g(x, t, z)\}, \end{aligned} \tag{3.3}$$

where the second equality comes from Property 1 in Lemma 3.1. Noting that $f(y, h, \theta_{(-t-h,0)}z) = f(y, h, \theta_{(-h,0)}\theta_{-t}z)$ and applying the definition of g to (3.3) above, we obtain

$$\begin{aligned} g(x, t + h, z) &= \sup\{y \geq 0 : f(y, h, \theta_{(-h,0)}\theta_{-t}z) \leq g(x, t, z)\} \\ &= g(g(x, t, z), h, \theta_{-t}z). \end{aligned}$$

The risk process and duality

Given a two-sided process $Z = \{Z_t : t \in \mathbb{R}\}$ taking values in $\boldsymbol{U}$ with $Z_0 = 0$, we define the *risk process* as $\{R_t[x] : t \geq 0\}$, the dual of the content process $V[y]$, with initial reserve $x \geq 0$. For a fixed x and $\forall t \geq 0$ we set

$$R_t[x] \stackrel{\text{def}}{=} g(x, t, Z). \tag{3.4}$$

From Lemma 3.1, Proposition 3.1, and the definition of $V_0^{(-t)}[y]$ we obtain the following properties of $R_t[x]$:

Property 1: $R_t[x] \geq y$ if and only if $V_0^{(-t)}[y] \leq x$.

Property 2: $R_t[x] = -\infty$ if and only if $V_0^{(-t)}(0) > x$.

Property 3: $R_{t+h}[x] = g(R_t[x], h, \theta_{-t}Z)$.

Via Property 3 and Kolmogorov's extension theorem, the process $\{R_t[x] : t \geq 0\}$ defines a probability measure on the space of extended-real-valued functions for each fixed $x \geq 0$. This process wanders around in the interval $[0, \infty)$, earning and losing money until either getting *ruined* (by jumping to the value $-\infty$) or becoming infinitely rich (by achieving the value $+\infty$), whichever happens first (but it is possible that neither happens).

Given that Z has stationary increments, we showed that $V_t[y] = V_0^{(-t)}[y]$ in law. This fact, combined with the above properties of $R_t[x]$, leads to the following key corollary.

Corollary 3.1. *Given that Z has stationary increments, let V have the steady-state distribution of the content process. Let $\tau(x) = \inf\{t \geq 0 : R_t[x] = -\infty\}$, the time of ruin for the risk process with $x \geq 0$. Then*

1. $\mathrm{P}(V_t[y] \leq x) = \mathrm{P}(R_t[x] \geq y),\ t \geq 0.$

2. $\mathrm{P}(V_t(0) > x) = \mathrm{P}(\tau(x) \leq t),\ t \geq 0.$

3. $\mathrm{P}(V > x) = \mathrm{P}(\tau(x) < \infty).$

Proof. Taking expectations in Property 1 yields item 1. Because the point $-\infty$ is absorbing, taking expectations in Property 2 yields item 2. Then, letting $t \to \infty$ in item 2, we get item 3 by monotonicity (Lemma 2.1).

This corollary allows us to investigate the one-point probabilities of any stochastically monotone process that is driven by a stationary-increment process, via the corresponding probabilities of its dual and *vice versa.* Below we use these correspondences to establish some previously known and some unknown equalities.

4. Applications

It is important at this juncture to recall that the risk process $\{R_t[x]\}$ is, by definition, constructed from the time reversal of a two-sided Z; $R_t[x] \stackrel{\text{def}}{=} \sup\{y \geq 0 : f(y, t, \theta_{(-t,0)}Z) \leq x\}$ (recall (3.4)).

Example 1: Inventory process

Here $U = \mathbb{R}$ and z is a left-continuous function with right limits taking $\mathbb{R}$ into $\mathbb{R}$ with $z_0 = 0$. (Right-continuity with left limits could be assumed instead. For simplicity, however, we shall deal only with the left-continuous case.) Let $f(y, \cdot, z)$ be defined as the Skorohod mapping Γ of the path $\{y + z_t : t \geq 0\}$, i.e.

$$f(y, t, z) = \Gamma_t(y + z) = y + z_t + l_t[y], \tag{4.1}$$

where

$$l_t[y] = \sup_{0 \leq s \leq t} \{-y + -z_s\}^+. \tag{4.2}$$

It is straightforward to verify that f satisfies Conditions (A1)–(A3) and that f is left-continuous in t. The mapping taking a path z to the path $v = \{f(y,t,z) : t \geq 0\}$ is historically called the reflection mapping (even though it is not in general a true reflection). The path v corresponds to what is sometimes called an *inventory process* and includes workload in single-server queues and in dam and storage models (see in particular [9], [14] and [24]). z is sometimes called the *netput* since it represents input minus potential output, and l is then called the *lost potential output*. For example, in a workload queueing context (with server working at rate 1), $z_t = a_t - t$, where a_t denotes the cumulative amount of work that arrives during $(0,t]$, and l_t is precisely the cumulative idle time of the server during $(0,t]$. When $\mathrm{E}(Z_1) < 0$ (negative drift case), it is well known that $\mathrm{P}(V < \infty) = 1$ (see Section 6 in [10]).

To construct the dual function $g(x,t,z)$, we first look at the ruin of the dual process. If $f(0,t,\theta_{(-t,0)}z) > x$, then, by definition (3.1), $g(x,t,z) = -\infty$. By the left-continuity of $f(0,s,z)$ in s, there must exist a time σ_c for each $c \in (0,x)$, such that $\sigma_c = \inf\{s \in (0,t) : f(0,u,\theta_{(-t,0)}z) \geq c, \forall u \in [s,t]\}$. Two things concerning these times are immediately clear: (1) for each c, $l_{\sigma_c}[0] = l_t[0]$, and (2) at the point $\sigma = \lim_{c \downarrow 0} \sigma_c$ either $f = 0$ or the right limit of f equals 0. We now define a sequence of times $\{s_n,\ n \in \mathbb{N}\}$ such that if $f(0,\sigma,\theta_{(-t,0)}z) = 0$, then $s_n = \sigma, \forall n$, and, if not, then $s_n \downarrow \sigma$ with $s_n > \sigma$ for each n. With this we use the recursion relation for f to obtain

$$f(0,t,\theta_{(-t,0)}z) = \lim_{n\to\infty} f(0,t-s_n,\theta_{(-t,0)}z) + z_0 - \lim_{n\to\infty} z_{s_n-t} > x,$$

which implies that $x + \lim_{n\to\infty} z_{s_n-t} < 0$. Therefore, there exists a time $s \in [\sigma,t)$ such that $x + z_{s-t} < 0$. Conversely, if $s \in [0,t)$ such that $x + z_{s-t} < 0$, then

$$f(0,t,\theta_{(-t,0)}z) \geq f(0,s,\theta_{(-t,0)}z) + z_0 - (z_{s-t} + x) + x > x.$$

If the dual is not ruined by time t, then $f(0,t,\theta_{(-t,0)}z) = z_0 - z_{-t} + \sup_{-t\leq s\leq 0}\{z_{-t} - z_s\}^+ \leq x$. We set $y(x) \equiv x - f(0,t,\theta_{(-t,0)}z) + \sup_{-t\leq s\leq 0}\{z_{-t} - z_s\}^+ = x - (z_0 - z_{-t})$. Then, because $\sup_{-t\leq s\leq 0}\{-y(x) + z_{-t} - z_s\}^+ = 0$,

$$f(y(x),t,\theta_{(-t,0)}z) = y(x) + z_0 - z_{-t}.$$

Therefore, $f(y(x),t,\theta_{(-t,0)}z) = x$. By the same argument $f(y,t,\theta_{(-t,0)}z) > x$ for all $y > y(x)$, and so $g(x,t,z) = y(x)$. Consequently,

$$g(x,t,z) = \begin{cases} -\infty, & \text{if } \min_{-t\leq s\leq 0} z_s < -x, \\ x + z_{-t}, & \text{if } \min_{-t\leq s\leq 0} z_s \geq -x. \end{cases}$$

The risk process $\{R_t[x] : t \geq 0\}$ evolves as an unrestricted netput process, $\{x + Z_{-t} : t \geq 0\}$, that starts at level x and then moves according to the reversed increments of Z. That is, $R_t[x] = x + Z_{-t}$ until (if possible) $x + Z_{-t}$ enters the interval $(-\infty, 0)$, after which it takes on value $-\infty$ forever. We conclude that

$$\mathrm{P}(\tau(x) < \infty) = \mathrm{P}\left(\inf_{t\geq 0} Z_{-t} < -x\right),$$

and from Corollary 3.1 we reach the well-known result that

$$\mathrm{P}(V > x) = \mathrm{P}\left(\inf_{t\geq 0} Z_{-t} < -x\right).$$

Thus V has the same distribution as the maximum of the reversed, negated increments.

Example 2: Regulated process with two barriers

We now assume the same model as in Example 1 but with an additional reflecting barrier at level $b > 0$, so that the inventory content is restricted to $[0, b]$. f is formally defined as the Skorohod mapping of the path $\{y + z_t : t \geq 0\}$ on to the interval $[0, b]$, i.e.

$$f(y, t, z) = y + z_t + l_t[y] - u_t[y],$$

where

$$l_t[y] = \sup_{0 \leq s \leq t} \{-y - z_s + u_s[y]\}^+, \tag{4.3}$$

$$u_t[y] = \sup_{0 \leq s \leq t} \{-b + y + z_s + l_s[y]\}^+. \tag{4.4}$$

l_t and u_t are non-negative and non-decreasing in t with $l_0(0) = u_0(0) = 0$. While l_t increases only when the inventory level is 0, u_t increases only when the inventory level is b. See pp. 22–24 in [14] for such details including the existence of $l_t[y]$ and $u_t[y]$ as functions only of $(y, \theta_{(0,t)} z)$ (the extension from continuous paths to left-continuous paths being straightforward). Verifying Conditions (A1)–(A3) is also straightforward (see, for example, [14], p. 24, Proposition 13 for (A3)).

Because f can only take values in $[0, b]$, the dual process can only take values in the set $[0, b] \cup \{-\infty\}$. In this case, $g(x, t, z)$ is defined as $\sup\{y \in [0, b] : f(y, t, \theta_{(-t,0)} z) \leq x\}$. From this definition it is clear that b is a fixed point for g, in that if $g(x, t, z) = b$, then $g(r, x, z) = b, \forall r > t$. But, as we shall see below, b is more than a fixed point; it is a 'sticky' point, meaning that, regardless of the continuity properties of g in r, if $\exists t > 0$ such that $\lim_{r \uparrow t} g(x, r, Z) = b$, then $g(x, r, Z) = b, \forall r \geq t$. We say that the risk process 'wins' if it hits b. The following proposition completely defines the dual function in this case.

Proposition 4.1. *Let $\rho_0(x) = \inf\{r \in (0, t] : z_{-r} + x < 0\}$, and for all $c < b$ let $\tau_c(x) = \inf\{r \in (0, t] : z_{-r} + x \geq c\}$ with $\inf\{\emptyset\} = \infty$. Then*

$$g(x, t, z) = \begin{cases} -\infty, & \text{if } \exists c < b, \text{ s.t. } \rho_0(x) < \tau_c(x), \\ x + z_{-t}, & \text{if } \exists c < b, \text{ s.t. } x + z_{-r} \in [0, c], \quad \forall r \in (0, t], \\ b, & \text{if } \forall c < b, \tau_c(x) < \rho_0(x). \end{cases} \tag{4.5}$$

Proof. We start, as in Example 1, by looking at the set of z paths for which g starting at x is ruined by time t, i.e. when $f(0, t, \theta_{(-t,0)} z) > x$. To get a handle on this set, we again define for each $c \in (0, x)$, $\sigma_c = \inf\{s \in (0, t) : f(0, u, \theta_{(-t,0)} z) \geq c, \forall u \in [s, t]\}$. The left-continuity of f in time implies that such times exists. Repeating the same steps as in Example 1, we see that there must exist a time $\sigma \in [0, t)$ such that $x + z_{\sigma - t} < 0$.

To establish that there must exist a $c < b$ in this case such that $x + z_s \leq c, \forall s \in (\sigma - t, 0)$, we first assume the opposite. Then, by the continuity properties of z, $\exists \tau \in (0, t - \sigma)$ such that either $x + z_{-\tau} \geq b$ or $\lim_{r \uparrow \tau} x + z_{-r} = b$. In the former case recursion yields

$$\begin{aligned} f(0, t, \theta_{(-t,0)} z) &= f(f(0, t - \tau, \theta_{(-t,0)} z), \tau, \theta_{(-\tau,0)} z) \\ &= f(0, t - \tau, \theta_{(-t,0)} z) + z_0 - z_{-\tau} \\ &\quad - \sup_{t - \tau \leq s \leq t} \{-b + f(0, t - \tau, \theta_{(-t,0)} z) + z_{s-t} - z_{-\tau}\}^+ > x. \end{aligned}$$

This last equation implies that $f(0, t-\tau, \theta_{(-t,0)}z) > b$, which is, of course, impossible. In the latter case we proceed similarly, noting that

$$\lim_{r\uparrow\tau}\left[f(0,t-r,\theta_{(-t,0)}z)+z_0-z_{-r}-\sup_{t-r\le s\le t}\{-b+f(0,t-r,\theta_{(-t,0)}z)+z_{s-t}-z_{-r}\}^+\right]$$
$$=\lim_{r\uparrow\tau} f(f(0,t-r,\theta_{(-t,0)}z),r,\theta_{(-r,0)}z)=f(0,t,\theta_{(-t,0)}z)>x.$$

This also implies that there exists some point $s\in[0,t]$ such that $f(0,s,\theta_{(-t,0)}z)>b$. Thus, there must be some $c<b$, for which $x+z_s<c$, $\forall s\in(\sigma-t,0)$.

To prove that $f(0,t,\theta_{(-t,0)}z)>x$, if $\exists\sigma\in[0,t)$ and $c<b$ such that $x+z_{\sigma-t}<0$ and $x+z_{s-t}<c$ for all $s\in[\sigma,t]$, we assume the opposite, i.e. that $f(0,t,\theta_{(-t,0)}z)\le x$, and draw a contradiction. If this is true, there exists $s'<t$ such that $u_s[0]$ is constant for all $s\in(s',t]$. So $\forall s\in(s',t]$

$$x\ge f(0,t,\theta_{(-t,0)}z)\ge f(0,s,\theta_{(-t,0)}z)+z_0-z_{s-t}.$$

This leads to the inequality

$$x+z_{s-t}\ge f(0,s,\theta_{(-t,0)}z)=z_{s-t}-z_{-t}+l_s[0]-u_s[0].$$

Thus, as s decreases, $f(0,s,\theta_{(-t,0)}z)$ can only cross above the path $x+z_{s-t}$ when $x+z_{s-t}=b$. But by assumption $f(0,s,\theta_{(-t,0)}z)<c, \forall s\in[\sigma,t]$. Therefore, $u_s[0]$ is constant $\forall s\in[\sigma,t]$ and

$$0>x+z_{\sigma-t}\ge f(0,\sigma,\theta_{(-t,0)}z).$$

This leads to $f(0,t,\theta_{(-t,0)}z)<0$, an impossibility. Therefore, we conclude that $g(x,t,z)=-\infty$ if and only if the path $x+z_{-r}$ dips below zero before it nears the set $[b,\infty)$ as r increases to t.

Next we look at the set of paths on which the dual process has 'won' by time t. From the definition of g we see that $g(x,t,z)=b$ if and only if $f(b,t,\theta_{(-t,0)}z)\le x$. An argument identical to the one given above for the ruin of g shows that $f(b,t,\theta_{(-t,0)}z)\le x \Leftrightarrow \tau_c(x)<\rho_0(x)$ for each $c<b$. In other words, $g(x,t,z)=b$ if and only if the path $x+z_{-r}$ either hits the set $[b,\infty)$ or comes infinitely close to this set. The proof of this is left to the interested reader.

To complete the construction of g, we need to determine the value of $g(x,t,z)$ when $f(0,t,\theta_{(-t,0)}z)\le x<f(b,t,\theta_{(-t,0)}z)$. From the above arguments we know that $x+z_{-r}\in[0,c]$, $\forall r\in[0,t]$ for some $c<b$. Setting $y(x)=x+z_{-t}$, we have $\forall s\in[0,t]$

$$f(y(x),s,\theta_{(-t,0)}z)=x+z_{s-t}+l_s(y(x))-u_s(y(x))=x+z_{s-t}.$$

The last equality comes from the fact that the path $\{y(x)+z_{s-t}-z_{-t} : s\in[0,t]\}$ never leaves the interval $[0,c]$. Thus, $f(y(x),t,\theta_{(-t,0)}z)=x$. To see that $y(x)$ is the largest y for which $f(y,t,\theta_{(-t,0)}z)\le x$, let us take $y\in(y(x),y(x)+b-c)$. Then $f(y,t,\theta_{(-t,0)}z)=y+z_0-z_{-t}>x$. Therefore, $g(x,t,z)=y(x)=x+z_{-t}$. This completes the construction of the dual process in terms of the paths of z and establishes Equation (4.5).

Given a left-continuous, stationary-increment process $Z:\mathbb{R}\mapsto\mathbb{R}$, set $\gamma_0(x)=\inf\{r\ge 0 : x+Z_{-r}<0\}$ and $\delta_c(x)=\inf\{r\ge 0 : x+Z_{-r}\ge c\}$. We then have the following.

Proposition 4.2. *Let V have the steady-state distribution corresponding to the above content function f driven by the process Z. Then*

$$\mathrm{P}(V>x)=\mathrm{P}(\gamma_0(x)<\delta_c(x),\ \textit{for some } c<b).$$

In words: $P(V > x)$ *is precisely the probability that the unrestricted reversed netput* $\{Z_{-t} : t \geq 0\}$ *dips below the level* $-x$ *before nearing the level* $b - x$.

If $\tau_b(x) = \lim_{c \uparrow b} \tau_c(x)$ a.s. for each fixed $x \in [0, b)$, as is the case when Z_t is continuous or when Z has independent as well as stationary increments, then the above equation simplifies to

$$P(V > x) = P(\gamma_0(x) < \delta_b(x)).$$

Example 3: Diffusions

Here we are interested in the dual to a reflected, 1-D diffusion restricted to $[0, \infty)$. (General questions concerning duality in the context of diffusions can be found in the notes of [1].) Let the drift coefficient $b(y)$ and the diffusion coefficient $a(y)$ for our diffusion be real-valued, continuous functions. Given certain conditions on b and a, we can describe the unrestricted diffusion $X[y]$ as the solution to the stochastic differential equation (SDE)

$$X_s[y] = y + \int_0^s b(X_r[y])\,\mathrm{d}r + \int_0^s \alpha(X_r[y])\,\mathrm{d}B_r, \quad \forall s \geq 0, \tag{4.6}$$

where $\{B_s : s \geq 0\}$ is a Brownian motion with $\alpha(y)$, sometimes called the dispersion coefficient, $= \sqrt{a(y)}$. We use the standard Itô definition for the stochastic integral above. To construct the reflected version V of this diffusion as a solution to an SDE system, we once more turn to the Skorohod mapping on $[0, \infty)$ defined by Equation (4.1). For all $t \geq 0$ let

$$Y_t[y] = y + \int_0^t b(V_s[y])\,\mathrm{d}s + \int_0^t \alpha(V_s[y])\,\mathrm{d}B_s, \tag{4.7}$$

$$V_t[y] = \Gamma_t(Y[y]) = Y_t[y] + \sup_{0 \leq s \leq t} \{-Y_s[y]\}^+. \tag{4.8}$$

Y is an unrestricted diffusion whose drift and diffusion parameters at time t depend on the value of $\Gamma_t(Y[y])$. V is our reflected diffusion corresponding to b and a. It moves as a normal diffusion when > 0, but is forced back into $[0, \infty)$ whenever it attempts to leave this interval. Given that for some $K > 0$

$$|b(x) - b(y)| + |\alpha(x) - \alpha(y)| \leq K|x - y|, \quad \forall x, y \in \mathbb{R},$$

there exists a unique, continuous, strong solution to Equation (4.7) and to Equation (4.8) for any Brownian motion path B (see [2], [12] for details). Existence and uniqueness of a strong solution to Equation (4.7) are, of course, only almost sure properties given any Brownian motion probability space. We can, however, simply throw away any Brownian path for which these properties do not hold, and thus dispense with the 'almost sure' restriction. We note that although it is clearly not necessary that $b(y)$ and $\alpha(y)$ be defined, let alone Lipschitz-continuous, for $y < 0$ in order to define Y and V, we can do so without loss of generality. As we shall see, this extension proves useful in defining the dual of V.

Now let $U = \mathbb{R}$ and $Z =$ a two-sided Brownian motion B. We set

$$f(y, t, B) = V_t[y].$$

Before we construct g, we need to show that f satisfies (A1)–(A3). By definition $f(y, t, B)$ satisfies (A1). To see that $f(y, t, B)$ is non-decreasing in y, we note that continuity in time implies that if $y_1 < y_2$, then either $V_t(y_1) < V_t(y_2)$, $\forall t \geq 0$, or $\exists \tau$ such that $V_\tau(y_1) = V_\tau(y_2)$.

In the latter case the strong uniqueness of the solution to Equation (4.8) implies that $V_t(y_1) = V_t(y_2), \forall t \geq \tau$. Thus, $f(y_1, t, B) \leq f(y_2, t, B)$. For any fixed B we also claim that $f(y, t, B)$ is continuous in y, $\forall t$. To see this, we note that if $Y^{(1)}$ and $Y^{(2)}$ are continuous functions, then $\|\Gamma(Y^{(1)}) - \Gamma(Y^{(2)})\|_t \leq 2\|Y^{(1)} - Y^{(2)}\|_t$ with $\|Y\|_t = \max_{0\leq s\leq t} |Y_s|$. Thus,

$$\begin{aligned}\mathrm{E}[\|V[y_1] - V[y_2]\|_t^2] &\leq 2\mathrm{E}[\|Y[y_1] - Y[y_2]\|^2] \\ &\leq C|y_1 - y_2|^2 + C'(1+t)\int_0^t \mathrm{E}[\|V[y_1] - V[y_2]\|_s^2]\,\mathrm{d}s. \end{aligned} \tag{4.9}$$

To obtain the second inequality above, we used Equation (4.7) together with the Cauchy–Schwartz inequality and Doob's martingale inequality. Applying Gronwald's inequality to line (4.9), we obtain $\mathrm{E}[\|V[y_1] - V[y_2]\|_t^2] \leq C|y_1 - y_2|^2$, where $C > 0$ depends only on t and K, the Lipschitz constant for $b(y)$ and $\alpha(y)$. This, combined with the Kolmogorov criterion, implies that there exists a version of V such that $V_s[\cdot]$ is a continuous function for each $s \in [0, t]$. Thus, (A2) is satisfied. Finally, it is clear that strong uniqueness implies that f satisfies (A3), the recursion relation.

In order to define $g(x, t, B)$ explicitly and uniquely in terms of the paths of B, we also assume that $\alpha \in C^1$ and the function $h(y) \stackrel{\text{def}}{=} \alpha(y)\alpha'(y)$ is Lipschitz-continuous and grows no faster than linearly. By analogy with Example 1 we would expect the dual process to be the time-reversed solution to Equation (4.6) with absorption below the level $x = 0$. This is, in fact, the case, and we state the following.

Proposition 4.3. *Let B be a Brownian motion path and let $f(y, t, B) = V_t[y]$, defined by Equation (4.8). Let $\{\tilde{B}_r : r \geq 0\} \stackrel{\text{def}}{=} \{B_{-r} : r \geq 0\}$ and $W[x]$ be the unique, strong solution to the SDE*

$$W_t[x] = x + \int_0^t [h(W_r[x]) - b(W_r[x])]\,\mathrm{d}r + \int_0^t \alpha(W_r[x])\,\mathrm{d}\tilde{B}_r. \tag{4.10}$$

Then

$$g(x, t, B) = \begin{cases} -\infty, & \text{if } \exists r \in [0, t] : W_r[x] < 0, \\ W_t[x], & \text{if } \forall r \in [0, t], W_r[x] \geq 0. \end{cases} \tag{4.11}$$

It is a standard result that, given the above assumptions on b and α, Equation (4.10) has a unique, strong solution $\{W_t[x] : t \geq 0\}$ that is continuous in t and is continuous and non-decreasing as a function of x, $\forall t$. We also note that by the same token Equation (4.6) has a unique, strong solution $\{X_t[y] : t \geq 0\}$ that, like V, is continuous in both t and y, as well as non-decreasing in y.

Before beginning the proof proper of our proposition, we give another definition of $W_t[x]$. Let us first define the non-Itô stochastic integral of a process $q(r)$, which is adapted to the past of the Brownian motion $\tilde{B}$, as

$$\int_0^t q(r) \circ \mathrm{d}\tilde{B}_r \stackrel{\text{def}}{=} \lim_{\epsilon\to 0} \sum_{n=1}^{N_\epsilon} q(r_n)(\tilde{B}_{r_n} - \tilde{B}_{r_{n-1}}), \tag{4.12}$$

where $\{r_n : n \leq N_\epsilon\}$ is an ϵ-mesh of $[0, t]$ with $r_i \leq r_j$ if $i \leq j$. Here the Brownian motion path is incremented backwards in time. Therefore, the integrand $q(r_n)$ and the increment

$\tilde{B}_{r_n} - \tilde{B}_{r_{n-1}}$ are no longer independent, as is the case in an Itô integral. This stochastic integral is not equal to the Itô integral; there is, however, a relationship. In particular, $W_t[x]$, the solution to Equation (4.10), is equivalent to the solution to the following SDE system:

$$W_t[x] = x - \int_0^t b(W_r[x])\,\mathrm{d}r + \int_0^t \alpha(W_r[x]) \circ \mathrm{d}\tilde{B}_r, \quad \forall t \geq 0, \tag{4.13}$$

where we use Equation (4.12) to define the integral w.r.t. $\tilde{B}$. Throughout the proof below we shall use this alternate definition of $W[x]$, noting that this SDE also must have a unique, continuous, strong solution for each fixed x and $\tilde{B}$. The reason for using this definition is the following: because both Equation (4.6) and Equation (4.13) have unique, strong solutions, if $W_t[x] = y$, then the path $\{W_{-s}[x] : s \in [-t, 0]\}$ satisfies the SDE

$$X_s^{(-t)}[y] = y + \int_{-t}^s b(X_u^{(-t)}[y])\,\mathrm{d}u + \int_{-t}^s \alpha(X_u^{(-t)}[y])\,\mathrm{d}B_u, \quad \forall s \in [-t, 0]. \tag{4.14}$$

To see this, we note that $\forall s \in [-t, 0]$

$$W_{-s}[x] = W_t[x] - (W_t[x] - W_{-s}[x]) \tag{4.15}$$

$$= y + \int_{-s}^t b(W_r[x])\,\mathrm{d}r - \int_{-s}^t \alpha(W_r[x]) \circ \mathrm{d}\tilde{B}_r \tag{4.16}$$

$$= y + \int_{-t}^s b(W_{-u}[x])\,\mathrm{d}u + \int_{-t}^s \alpha(W_{-u}[x])\,\mathrm{d}B_u. \tag{4.17}$$

In the last line above we set $u = -r$. Because we have reversed the direction of time, the stochastic integral in the last line is a standard Itô integral. Thus, $W_{-s}[x] = X_s^{(-t)}[y], \forall s \in [-t, 0]$. In other words, $W[x]$ solves the time-reversal of the SDE that $X^{(-t)}[y]$ solves. By the same logic, if $X_0^{(-t)}[y] = x$, then $X_{-r}^{(-t)}[y] = W_r[x], \forall r \in [0, t]$.

Proof. To construct the dual function, let us once more start by looking at the ruin of $g(x, t, B)$. If $V_0^{(-t)}[0] \equiv f(0, t, \theta_{(-t,0)}B) > x$ (and g is ruined), there exists $\sigma \in [-t, 0)$ such that σ is the last time $s < 0$ that $V_s^{(-t)}[0] = 0$. Therefore,

$$V_0^{(-t)}[0] = f(V_\sigma^{(-t)}[0], -\sigma, \theta_{(\sigma,0)}B) = f(0, -\sigma, \theta_{(\sigma,0)}B) = V_0^{(\sigma)}[0].$$

Note that although σ is a random time when considered a function on the path space of B, it is a fixed time for any particular path B. Therefore, the recursion relation can be applied at time σ, as well as any truly constant time s.

Because $V_s^{(\sigma)}[0]$ (denoted $V_s^{(\sigma)}$) $> 0, \forall s \in (\sigma, 0]$, we have

$$V_s^{(\sigma)} = \int_\sigma^s b(V_u^{(\sigma)})\,\mathrm{d}u + \int_\sigma^s \alpha(V_u^{(\sigma)})\,\mathrm{d}B_u.$$

Given this, with $W_r[x]$ defined by Equation (4.13), we claim that $W_{-\sigma}[x] < 0$. If not, then because $W[x]$ starts below $V_0^{(\sigma)}$, $\exists r' \in (0, -\sigma]$ such that $W_{r'}[x] = V_{-r'}^{(\sigma)}$. Because $W[x]$ solves the time-reversal of the SDE that $V^{(\sigma)}$ solves, $W_r[x] = V_{-r}^{(\sigma)}, \forall r \in [0, r']$. But this contradicts the fact that $W_0[x] < V_0^{(\sigma)} = f(0, t, \theta_{(-t,0)}B)$. Therefore, $W_{-\sigma}[x] < 0$ (see Figure 1).

Conversely, we want to show that if $W_r[x] < 0$ for some $r \in (0, t]$, then $f(0, t, \theta_{(-t,0)}B) > x$, and, thus, $g(0, t, B) = -\infty$. So we shall assume that (1) $W_r[x] < 0$ for some $r \in (0, t]$ and (2) $V_0^{(-t)}[0] = f(0, t, \theta_{(-t,0)}B) \leq x$ and draw a contradiction.

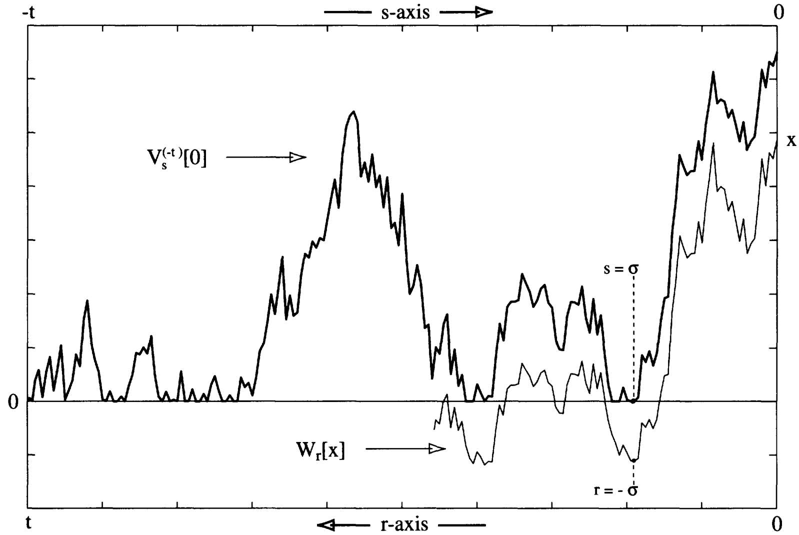

FIGURE 1: An illustration of the fact that if $V_0^{(-t)} > x$, then $\exists \sigma \in [-t, 0]$ such that $W_{-\sigma}[x] < 0$. In this figure s is the 'forward' time variable and r $(= -s)$ is the 'backward' time variable.

First, we note that, regardless of our assumptions, there must exist an $x' > 0$ such that $W_r(x') > 0$ for all $r \in [0, t]$. If not, then the monotonicity of W in x for every r implies that $\exists \tau \in [0, t]$ such that $W_\tau[x] \leq 0$ for all x. If $\{X_s^{(-\tau)}[y], s \in [-\tau, 0]\}$ satisfies

$$X_s^{(-\tau)}[y] = y + \int_{-\tau}^{s} b(X_u^{-\tau}[y])\,\mathrm{d}u + \int_{-\tau}^{s} \alpha(X_u^{-\tau}[y])\,\mathrm{d}B_r,$$

then $X_0^{-\tau}[0] = \infty$. This follows from the fact that $X_s^{(-\tau)}[W_\tau[x]] = W_{-s}[x], s \in [-\tau, 0]$ and the monotonicity of the function $X_0^{(-\tau)}[\cdot]$. But because $X_s[y]$ is continuous in s, $X_0^{(-\tau)}[0]$ cannot equal ∞, and so there must be an x' with $W_r[x'] > 0, \forall r \in [0, t]$. Given assumption (1), x' must also be greater than x.

Setting $y' \equiv W_t(x')$, we see that $W_{-s}[x']$ must equal $V_s^{(-t)}[y'], \forall s \in [-t, 0]$. This follows from the fact that the path $\{W_{-s}[x'], s \in [-t, 0]\}$ satisfies Equation (4.14) and stays above 0 for all $s \in [0, t]$. Thus, $\{W_{-s}[x'], s \in [-t, 0]\}$ satisfies Equation (4.8) with time shifted backwards by t units.

Given assumption (2), we set $y(x) = \sup\{y : V_0^{(-t)}[y] = x\}$. Because $x' > x$, $y(x)$ must be less than y'. Then, for any $y \in (y(x), y']$, $V_s^{(-t)}[y] > 0, \forall s \in [-t, 0]$. If not, then $\exists s'$ such that $V_{s'}^{(-t)}[y] = 0$, which $= V_{s'}^{(-t)}[0]$ by monotonicity. Recursion then implies that $V_s^{(-t)}[y] = V_s^{(-t)}[0], \forall s \geq s'$. But this contradicts the fact that $V_0^{(-t)}[y] > x \geq V_0^{(-t)}[0]$. Therefore, $V^{(-t)}[y] = X^{(-t)}[y]$, where $X^{(-t)}[y]$ satisfies Equation (4.14). Because $V_s^{(-t)}[y] > 0, \forall s \in [0, t]$ and $\forall y \in (y(x), y']$, the continuity of X in y for each s then implies that $X_s^{(-t)}[y(x)] \geq 0$ for all $s \in [-t, 0]$. Thus, $V_s^{(-t)}[y(x)] = X_s^{(-t)}[y(x)]$ for all $s \in [-t, 0]$, as well. But if this is true, then $X_0^{(-t)}[y(x)] = x$, which implies that $W_r[x] = X_{-r}^{(-t)}[y(x)], \forall r \in [0, t]$. This leads to the conclusion that $W_r[x] \geq 0, \forall r \in [0, t]$, which contradicts our first assumption that $W_r[x] < 0$ for some $r \in (0, t]$. Thus, $f(0, t, \theta_{(-t,0)}B)$ must be greater than x in this case, proving that $g(x, t, B)$ is ruined if and only if $W_r[x] < 0$ for some $r \in (0, t]$.

Above we showed that if $f(0,t,\theta_{(-t,0)}B) \le x$, there must then exist a path $\{X_s^{(-t)}[y(x)], s \in [-t,0]\}$, satisfying the SDE of Equation (4.14) and starting at some $y(x) \ge 0$ with $X_0^{(-t)}[y(x)] = x$ such that $\forall s \in [0,t]$, $X_s[y(x)] \ge 0$. This again implies that $V_s^{(-t)}[y(x)] = X_s^{(-t)}[y(x)] = W_{-s}[x], \forall s \in [-t,0]$. Therefore, if $f(0,t,\theta_{(-t,0)}B) \le x$, $g(x,t,B) = y(x) = W_t[x]$. This completes the construction of the dual of a reflected diffusion.

Remark. One might also be interested in the dual of a diffusion (with drift $b(y)$ and diffusion coefficient $a(y)$) that has two reflecting barriers, one at $y = 0$ and the other at $y = c > 0$. One can define such a process uniquely, as in Example 2, through the use of the Skorohod mapping on the interval $[0,c]$. Similar to Example 3, the dual in this case is a diffusion with drift $a'(x)/2 - b(x)$ and diffusion coefficient $a(x)$. This diffusion, like the dual in Example 3, is absorbed ($= -\infty$) if it dips below $x = 0$, but now if it ever hits the set level $x = c$, it is stuck there forever. The proof of this, which we leave to the interested reader, is a straightforward combination of the proofs of Propositions4.1 and 4.3.

Stability conditions for reflected diffusions

An interesting question as yet unaddresssed in this paper is that of the stability of our content processes. It is often useful to know whether or not there exists a stationary distribution associated with the transition function of $V[y]$. Answering this question in the case of 1-D reflected diffusion processes on $[0,\infty)$ is quite straightforward. Using Corollary 3.1 and the work of Pinsky [19] (among others), we are able to find conditions on $b(x)$ and $\alpha(x)$ that ensure the stability of the content process $V[0]$ (i.e. $\mathrm{P}(V_t^* < \infty) = 1$ where V_t^* is the stationary version of $V[0]$ defined by Equation (2.7)). These conditions are not only sufficient to ensure the stability of $V[0]$ and, thus, the existence of a steady-state distribution associated with the transition function of $V[y]$, but are also necessary.

Proposition 4.4. *Assuming that $\alpha(x)$, $b(x)$ and $h(x) = \alpha(x)\alpha'(x)$ are all globally Lipschitz-continuous and that $\alpha(x) > 0, \forall x \ge 0$, the content process V, a diffusion reflected at $x = 0$, has a non-trivial steady-state version if and only if*

$$H(x) \stackrel{\text{def}}{=} \int_0^x \alpha^{-2}(y) \exp\left(2\int_0^y \frac{b(z)}{\alpha^2(z)}\,\mathrm{d}z\right)\mathrm{d}y \tag{4.18}$$

is a bounded function for $x \in [0,\infty]$. Furthermore, the steady-state distribution is given by

$$\mathrm{P}(V_t^* \le x) = H(x)/H(\infty).$$

Proof. Let $R[x]$ again be the dual of $V[y]$ starting at x. Let $\tau(x) = \inf\{t \ge 0 : R_t[x] < 0\}$ and let the r.v. V have the steady-state distribution of the content process. By Corollary 3.1

$$\mathrm{P}(V \le x) = \mathrm{P}(\tau(x) = \infty).$$

Thus, for V to have a non-trivial distribution there must be a positive probability that $R[x]$ heads off to infinity before it hits zero for some $x > 0$. Using Theorem 5.1.1 in [19], we see that this transience will occur if and only if $H(\infty) < \infty$ and that

$$\mathrm{P}(\tau(x) = \infty) = H(x)/H(\infty).$$

Given our assumptions on $\alpha(x)$ and $b(x)$, diffusions reflected at both 0 and at $c > 0$ are always stable. By trivially extending the above proposition to cover such content processes, we see that the steady-state distribution associated with these Markov processes is given by

$$P(V \leq x) = H(x)/H(c), \quad \forall x \in [0, c].$$

Example 4: Storage process with general release function

Assume V_t is a non-negative storage process that has input A_t with stationary increments ($A_0 = 0$) and that has release rate $b(y)$ when $V_t = y$. To define this process, we once more turn to the Skorohod mapping on $[0, \infty)$. We define V as the output of the reflected, integral-equation system

$$\begin{aligned} Y_t[y] &= y + A_t - \int_0^t b(V_s[y])\,\mathrm{d}s, \quad \forall t \geq 0 \text{ with } y \geq 0, \\ V_t[y] &= \Gamma_t(Y[y]) = Y_t[y] + \sup_{0 \leq s \leq t} \{-Y_s[y]\}^+. \end{aligned} \tag{4.19}$$

Y is, in some sense, a dummy function that simply facilitates the calculation of the content process V. If, however, we view $V_t[y]$ as a storage process, then $V_t[y] - Y_t[y]$ is the amount of supplies that was ordered up to time t and could not be delivered due to the fact that the warehouse was empty. In this storage context it is usual to assume that the increments of A are not only stationary, but non-negative and that the release rate $b(y)$ is a non-negative function for $y \geq 0$. These conditions, however, are unnecessary for the construction of V and its dual R. Below we will assume only that A has stationary increments and is a real-valued, left-continuous process and that $b(y)$ is a real-valued, Lipschitz-continuous function with a global Lipschitz constant $K > 0$.

Another context in which such processes arise is in queueing theory. Here V_t may denote workload, where A_t is the accumulated customer service time that jumps by the amount $S_n \geq 0$ at time t_n (customer arrival time), with $\{(t_n, S_n) : n \geq 0\}$ forming a time-stationary simple marked point process, $0 < t_0 < t_1 < \cdots$ (see [23]).

Our objective here is to deduce the evolutionary form of the risk process $\{R_t[x],\ t \geq 0\}$. In particular, we claim that

$$R_t[x] = \begin{cases} -\infty, & \text{if } W_r[x] < 0 \text{ for some } r \in [0, t], \\ W_t[x], & \text{if } W_r[x] \geq 0, \forall r \in [0, t], \end{cases} \tag{4.20}$$

where

$$W_t[x] = x + A_{-t} + \int_0^t b(W_r[x])\,\mathrm{d}r, \quad \forall t \geq 0. \tag{4.21}$$

This is very nice, for it tells us that the risk process before getting ruined evolves as follows: starting with reserve x, money flows *in* with a premium rate $b(x)$ when $R_t = x$ and money flows *out* according to the time reversal of A. (Remember that if A_t is a non-decreasing function, then A_{-t} is a non-increasing function.)

This result is well known in a variety of special cases such as in the $M/G/1$ queue (see [7] and [15]). It is also known to hold in somewhat more complicated examples (see, in particular, [5]).

To prove our claim, we first need to show that not only does the function $f(y, t, A) \stackrel{\text{def}}{=} V_t[y]$ satisfy (A1)–(A3) but, more fundamentally, that there exists a unique solution V to system (4.19) at all. The latter statement, however, is easy to show. Because $b(y)$ is a globally Lipschitz function and the Skorohod mapping Γ_t is also Lipschitz under the sup norm on the space of left-continuous functions, standard Picard-iteration/Gronwald-inequality methods yield a unique, left-continuous solution to this reflected, integral equation for all $t \geq 0$, given any fixed, left-continuous, input function A.

By the definition of Γ_t, f satisfies (A1). Defining as before $\|Y\|_t = \max_{0\leq s\leq t} |Y_s|$, we have for any $y_1, y_2 \geq 0$ and $t \geq 0$

$$\begin{aligned}\|V[y_2] - V[y_1]\|_t &\leq 2\|Y[y_2] - Y[y_1]\|_t \\ &\leq 4|y_2 - y_1| + 4 \sup_{0\leq s\leq t} \left\{ \int_0^s |b(V_u[y_2]) - b(V_u[y_1])| \,\mathrm{d}u \right\} \\ &\leq 4|y_2 - y_1| + 4K \int_0^t \|V[y_2] - V[y_1]\|_s \,\mathrm{d}s.\end{aligned}$$

This last inequality implies, via Gronwald's inequality, that $\|V[y_2] - V[y_2\|_T \leq C(T, K)|y_2 - y_1|$, which establishes the continuity of the function $f(\cdot, t, A)$ simultaneously for all t. To prove that $f(y, t, A)$ is also non-decreasing in y, we note that

$$V_t[y_2] - V_t[y_1] = y_2 - y_1 + \int_0^t [b(V_s[y_2]) - b(V_s[y_1])] \,\mathrm{d}s$$

is a continuous function in t. Therefore, if $y_2 > y_1$, either $V_t[y_2] > V_t[y_1]$, $\forall t \geq 0$ or $\exists \sigma \geq 0$ such that $V_\sigma[y_2] = V_\sigma[y_1]$. In the latter case the uniqueness of the solution to system (4.19) implies that $V_t[y_2] = V_t[y_1]$ for all $t \geq \sigma$. Therefore, f satisfies Condition (A2). Finally, we note that the uniqueness of the solution to system (4.19) also immediately establishes the recursion equation (A3) for $f(y, t, A)$.

Now that we have proved that $f(y, t, A)$ is a monotone, recursive function, we can get down to the business at hand, establishing that R, defined in Equation (4.20), is truly the dual of V. First, let the path $\{X_s^{(-t)}[y],\ s \in [-t, 0]\}$ be defined by the integral equation

$$X_s^{(-t)}[y] = y + A_s - A_{-t} - \int_{-t}^s b(X_u^{(-t)}[y]) \,\mathrm{d}u, \quad \forall s \in [-t, 0]. \tag{4.22}$$

For every y, $X^{(-t)}[y]$ is the unique, left-continuous solution to this equation. (Again Picard-iteration/Gronwald-inequality methods establish existence and uniqueness for the solution to this equation.) By the same logic that we used above to prove that $V_s[y]$ is continuous and non-decreasing in y, $X_s^{(-t)}[y]$ is also continuous and non-decreasing in y.

With this we claim that if $X_0^{(-t)}[y] = x$, then the path $\{X_{-r}^{(-t)}[y],\ r \in [0, t]\}$ solves Equation (4.21), i.e. that $X_{-r}^{(-t)}[y] = W_r[x]$, $\forall r \in [0, t]$. To see this, note that

$$\begin{aligned}X_{-r}^{(-t)}[y] &= X_0^{(-t)}[y] - (X_0^{(-t)}[y(x)] - X_{-r}^{(-t)}[y]) \\ &= x + A_{-r} - \int_{-r}^0 b(X_s^{(-t)}[y]) \,\mathrm{d}s \\ &= x + A_{-r} + \int_0^r b(X_{-u}^{(-t)}[y]) \,\mathrm{d}u.\end{aligned}$$

Thus, $X_{-r}^{(-t)}[y]$ solves Equation (4.21) and, by the uniqueness of the solution to this equation, equals $W_r[x]$. Conversely, if $W_t[x] = y$, then $W_{-s}[x] = X_s^{(-t)}[y]$, $\forall s \in [-t, 0]$.

To complete the proof of Equation (4.20), we refer the reader back to the proof of Proposition 4.3. At this point in our analysis the diffusion and storage cases are almost identical, and we have used similar notation in both to emphasize this. The only difference of any note is that our storage process is only left-continuous. However, this change is a mere technicality and affects very little in the proof of Proposition 4.3. Invoking that proof establishes our equation for R, the dual of V.

Example 5: Jump/diffusion processes

Our last application concerns the dual processes of non-negative, left-continuous, homogeneous Markov processes with diffusion and jump components. For such a jump/diffusion process X its infinitesimal generator $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{A}f(y) &\stackrel{\text{def}}{=} \lim_{t\to 0} t^{-1}\mathrm{E}[f(X_t[y]) - f(y)] \\ &= b(y)f'(y) + \tfrac{1}{2}a(y)f''(y) + \lambda(y)\int_0^\infty [f(x) - f(y)]Q(y, \mathrm{d}x) \qquad (4.23)\end{aligned}$$

for any $f \in C_0^2[0, \infty)$ with $f'(0) = 0$. In the above equation $b(y)$ and $a(y)$ are the drift and diffusion coefficients of X, respectively. $\lambda(y)$ is the jump intensity of X, when $X_t = y$, and we assume that $\sup_{y\geq 0}\lambda(y) = \lambda < \infty$. The jump measure $Q(y, A)$ with A a Borel set in $[0, \infty]$ gives the probability that, conditioned on $X_t = y$ and t being a jump point, X jumps from y into the set A. In other words, if A is a closed set in $[0, \infty]$ with $y \notin A$, then

$$\lambda(y)Q(y, A) = \lim_{t\to 0} t^{-1}\mathrm{P}[X_t[y] \in A].$$

The infinitesimal generator $\mathcal{A}$ of the Markov process X uniquely determines the process' probability distribution on the space of left-continuous functions. Thus, the above discussion describes X in probability; we, however, need a pathwise description of this process if we are to construct its dual. We also need to put restrictions on the measure Q to ensure that X is stochastically monotone. To acomplish both of these ends, we define a new jump measure $P_J(y, \cdot)$ on the Borel sets of $[0, \infty]$ as follows:

$$P_J(y, A) = \begin{cases} \dfrac{\lambda(y)}{\lambda}Q(y, A), & \text{if } y \notin A, \\ 1 - \dfrac{\lambda(y)}{\lambda}, & \text{if } A = \{y\}. \end{cases} \qquad (4.24)$$

We can define the 'jump' times for X as the event times of a Poisson process with intensity λ, given that the probability that X jumps from y into any Borel set A at such an event time equals $P_J(y, A)$. We put 'jump' in quotes because if $\lambda(y) < \lambda$, there is a positive probability that X will not actually jump at an event time t given $X_t = y$. By defining the jumps of X in this way, we have made the 'jump' times independent of the particular path of X. But clearly the jump intensities and jump probabilities at any point y are unaltered because $\lambda P_J(y, A) = \lambda(y)Q(y, A)$ if $y \notin A$. With this new jump measure it is easy to see what conditions are sufficient for X to be stochastically monotone and, in general, to satisfy (A1)–(A3). We assume that the drift and diffusion coefficients b and a satisfy the same conditions that they did in Example 3. For the jump component of X we shall require that $P_J(y, [0, x])$ be *right-continuous* and *non-decreasing* in x (an obvious necessity) and be *left-continuous* and *non-increasing* in y. The left-continuity of P_J in y is necessary if we are to contruct a process X that satisfies (A2), and the fact that P is non-increasing in y is necessary if X is to be stochastically monotone. With this we can now construct a pathwise version of X that is stochastically monotone and define our recursive function f for this case.

Let $\boldsymbol{U} = \mathbb{R} \oplus \mathbb{M}$, where $\mathbb{M}$ = the space of signed point measures on $[0, 1]$ normed by $\|\mu\| = \int_0^1 |\mu|(\mathrm{d}u)$, the total mass of μ. Let $Z_t \stackrel{\text{def}}{=} (B_t, N_t(\mathrm{d}u))$, where B is a Brownian motion on $\mathbb{R}$ with $B_0 = 0$ and where N is a Poisson point process, taking values in $\mathbb{M}$ with characteristic measure $\nu(\cdot) = \lambda l(\cdot)\,\mathrm{d}t$ ($l(\cdot)$ indicating Lebesgue measure on $[0, 1]$) and with N_0 = the zero element in $\mathbb{M}$. In other words, for any Borel set $E \subseteq [0, 1]$ if $t > s$, $P(N_t(E) - N_s(E) = k) = \mathrm{e}^{-\lambda l(E)(t-s)}[\lambda l(E)(t-s)]^k/k!$ for $k = 0, 1, \ldots$. If $[s_1, t_1] \times E_1 \cap [s_2, t_2] \times E_2 = \emptyset$, $N_{t_1}(E_1) - N_{s_1}(E_1)$ is independent of $N_{t_2}(E_2) - N_{s_2}(E_2)$. We take B to be a continuous Brownian motion and N to be a left-continuous process in t with the above norm on $\mathbb{M}$. Next, we set

$$F(y, u) \stackrel{\text{def}}{=} \inf\{x \geq 0 : P_J(y, [0, x]) \geq u\}. \tag{4.25}$$

We define V (our pathwise definition of the Markov process X) by the functional SDE

$$\begin{aligned} Y_t[y] &= y + \int_0^t b(V_s[y])\,\mathrm{d}s + \int_0^t \alpha(V_s[y])\,\mathrm{d}B_s + \int_0^t \int_0^1 (F(V_s[y], u) - V_s[y])\,\mathrm{d}N_s(\mathrm{d}u), \\ V_t[y] &= \Gamma_t(Y[y]) = Y_t[y] + \sup_{0 \leq s \leq t} \{-Y_s[y]\}^+, \end{aligned} \tag{4.26}$$

where $\Gamma_t(\cdot)$ is the Skorohod mapping on to $[0, \infty)$. An easy way to see that a strong solution to this SDE exists and is unique is to note that in between the event times of N, V evolves as in Example 3. Then, at an event time τ when $N_{\tau^+}(\mathrm{d}u) - N_\tau(\mathrm{d}u) = \delta(u - u_0)$, V jumps from $V_\tau[y]$ to $F(V_\tau[y], u_0)$, i.e. $V_{\tau^+}[y] = F(V_\tau[y], u_0)$. The process then begins anew at time τ^+ with initial condition $V_{\tau^+}[y]$.

We still must show that the infinitesimal generator of this Markov process is given by Equation (4.23). First, it is clear that the diffusion component of the infinitesimal generator has the correct drift and diffusion coefficients. To check that the jump component of $V[y]$ yields the correct term in the infinitesimal generator, we note that for $x < y$

$$\lim_{t \to 0} t^{-1}\mathrm{P}(V_t[y] \leq x) = \lambda \mathrm{P}(F(y, U) \leq x),$$

where U is a uniform random variable on $[0, 1]$, and for $x > y$

$$\lim_{t \to 0} t^{-1}\mathrm{P}(V_t[y] > x) = \lambda \mathrm{P}(F(y, U) > x).$$

By Definition (4.25) we see that $F(y, u) \leq x$ if and only if $u \leq P_J(y, [0, x])$. Therefore, $P(F(y, U) \leq x) = P_J(y, [0, x])$. Given the definition of P_J, we immediately obtain the equivalence in law of V and the Markov process X.

Setting $f(y, t, Z) = V_t[y]$, we now need to show that f satisfies (A1)–(A3). Non-negativity is clearly satisfied. To see that f is non-decreasing and left-continuous in y, we first note that up until the first jump time $V_t[y]$ is continuous and non-decreasing in y as in Example 3. At the first jump time $\tau_1 \geq 0$, $V_{\tau_1}[y] \to F(V_{\tau_1}[y], u_1)$. Definition (4.25) and the properties of

$P_J(y, [0, x])$ imply that $F(y, u)$ is non-decreasing and left-continuous in y and in u. Because the composition of two left-continuous, non-decreasing functions is left-continuous and non-decreasing, V still satisfies (A2) after the jump. Iteration then shows that this condition holds for all t. The recursion relation for f follows from the strong uniqueness of Equation (4.26).

With f defined we can look for the evolutionary form of the dual process $g(x, t, Z)$. Given any particular $Z = (B, N)$, we have a sequence of pairs $\{(\tau_n, u_n), n = 1, 2, \dots\}$, where τ_n is the nth event time as we go backwards in time from $t = 0$ and u_n is the location of the point at τ_n, i.e. $N_{\tau_n^+}(\mathrm{d}u) - N_{\tau_n}(\mathrm{d}u) = \delta(u - u_n)$. Now for $r \in [0, -\tau_1)$, $g(x, r, Z)$ is identical to the dual in Example 3, because there is no jump. At $t = -\tau_1$ we have

$$g(x, -\tau_1, Z) = \sup\{y \geq 0 : f(y, -\tau_1 - r, \theta_{\tau_1} Z) \leq g(x, r, Z)\}, \quad \forall r \in [0, -\tau_1)$$

by the inverse recursion relation for g. Therefore, if $g(x, r, Z) = -\infty$ for any r in this interval, then $g(x, -\tau_1, Z) = -\infty$, and we are done. If not, then $g(x, r, Z) \geq 0, \forall r \in [0, -\tau_1)$. Thus, $\lim_{r \uparrow -\tau_1} g(x, r, Z) = g^- \geq 0$ by the continuity of g in this interval. Also, $\lim_{r \uparrow -\tau_1} f(y, -\tau_1 - r, \theta_{\tau_1} Z) = F(y, u_1)$. From this we get

$$g(x, -\tau_1, Z) = \sup\{y \geq 0 : F(y, u_1) \leq g^-\}.$$

Therefore, at jump point $-\tau_1$, g jumps from g^-, the value of g just prior to $-\tau_1$, to

$$G(g^-, u_1) \stackrel{\text{def}}{=} \sup\{y \geq 0 : F(y, u_1) \leq g^-\}.$$

Because $F(y, u)$ is left-continuous and non-decreasing in y and u, $G(x, u)$ is right-continuous and non-decreasing in x and left-continuous and non-increasing in u, taking $[0, \infty] \times [0, 1]$ to $\{-\infty\} \cup [0, \infty]$. Now that we have the value of g at time $-\tau_1$, iteration of the above argument yields the evolutionary form of $g(x, t, Z)$ for all t. Therefore, the dual process $R_t[x]$ is a jump/diffusion process with absorption below zero and right-continuous jumps and is described by the equation

$$R_t[x] = \begin{cases} W_t[x], & \text{if } W_r[x] \geq 0, \forall r \in [0, t], \\ -\infty, & \text{if } \exists r \in [0, t] \text{ s.t. } W_r[x] < 0, \end{cases} \tag{4.27}$$

where

$$W_t[x] = x + \int_0^t [h(W_r[x]) - b(W_r[x])]\,\mathrm{d}r + \int_0^t \alpha(W_r[x])\,\mathrm{d}\tilde{B}_r + \int_0^t \int_0^1 (G(W_{r^-}[x], u) - W_{r^-}[x])\,\mathrm{d}N_{-r}(\mathrm{d}u),$$

with $h(y) = \alpha'(y)\alpha(y)$ and $\tilde{B}_r = B_{-r}, \forall r \geq 0$.

Finally, we note that $\mathrm{P}(G(x, U) \geq y)$, the probability that at an event time the process R jumps from x to the set $[y, \infty]$, equals $\sup\{u \in [0, 1] : G(x, u) \geq y\}$ because U is a uniform r.v. on $[0, 1]$ and G is non-increasing in u. By duality this should equal $P_J(y, [0, x])$. This is easily shown:

$$\begin{aligned} \sup\{u \in [0, 1] : G(x, u) \geq y\} &= \sup\{u \in [0, 1] : \sup\{w \geq 0 : F(w, u) \leq x\} \geq y\} \\ &= \sup\{u \in [0, 1] : \forall w < y, F(w, u) \leq x\} \\ &= \sup\{u \in [0, 1] : F(y, u) \leq x\} = P_J(y, [0, x]). \end{aligned}$$

The last line uses the fact that $F(w, u)$ is non-decreasing and left-continuous in w.

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