Signal to Interference Cells of a Spatial Point Process

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Structure of the Talk

Capacity of a Gaussian Channel

- Capacity in bits/second of the Hertzian channel between a transmitter and a receiver is given by Shannon's second theorem in the Gaussian case:

$$\theta = B \log(1 + \frac{R}{W + I})$$

  - B bandwidth of the frequencies used by the channel;
  - R power with which the signal is received;
  - W power of thermal noise at the receiver;
  - I power of interference (other signals at the receiver).

- Hence $\theta \geq K$ iff SINR$=\frac{R}{W+I} \geq T$, where the mapping $K = f(T)$ is determined by B.

SINR Cell of a Spatial Point Process

- $\Phi = \{X_i, (S_i, T_i)\}$ marked point process
- $\{X_i\}$ points of the p.p. on $\mathbb{R}^d$: location of transmitters
- (S_i, T_i): mark of point X_i: $(S_i, T_i) \in \mathbb{R}^+ \times \mathbb{R}^+$

SINR Cell attached to point X_0:

$$C_0(\Phi, W) = \{y : \frac{S_0 l(y - X_0)}{W + \kappa I_\phi(y)} \geq T_0\}$$

- S_i: power of transmitter i; W: power of thermal noise;
- $l(\cdot)$ attenuation function or path loss; κ: orthogonality factor.
- $I_\phi(y) = \Sigma_{i \neq 0} S_i l(y - X_i)$: power of interference at y,

C_0: set of locations y where channel from X_0 can sustain a bit rate of $f(T_0)$.

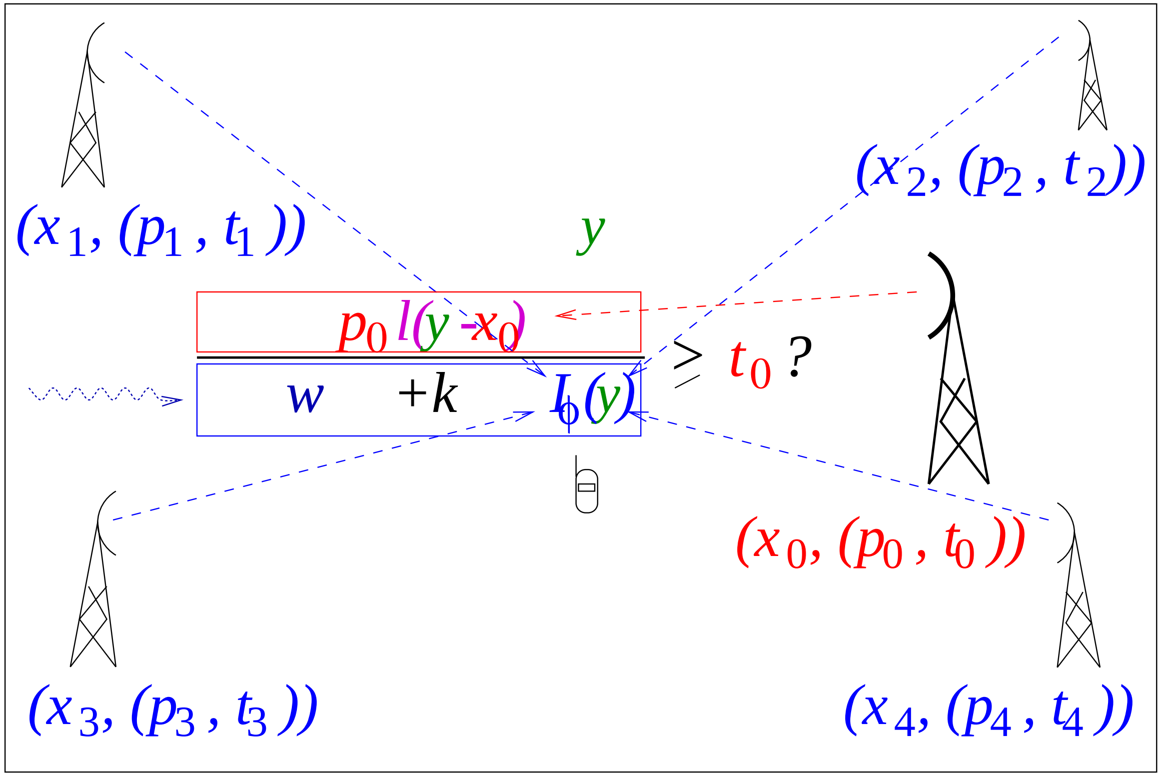
TYPICAL EXAMPLES OF ATTENUATION FUNCTIONS

- Unbounded support examples (for some $a, \beta, R > 0$)

1. $l(z) \leq a|z|^{-\beta}$
2. $l(z) \leq a(1 + |z|)^{-\beta}$
3. $l(z) = a(\max(|z|, R)^{-\beta}$

- Bounded support example: $l \leq M$, $l(z) = 0$ for $|z| > d$

MOTIVATION EXAMPLE 1: DOWNLINK IN CDMA



CDMA Basic Principles

- Large shared spectrum;
- All users transmit simultaneously (in contrast to what happens in FDMA or TDMA) which creates some global interference signal;
- Transmission i is allocated a spread spectrum signature process c_i built from a pseudo random sequence, which is used to modulate its signal; c_i is used by the receiver to extract the signal of transmission i. from the global signal.
- The orthogonality factor κ is smaller when codes are longer/more exactly orthogonal sequences and when propagation has less reflections/multiple paths.

VARIANTS OF BASIC CDMA MODEL

- Directional antennas

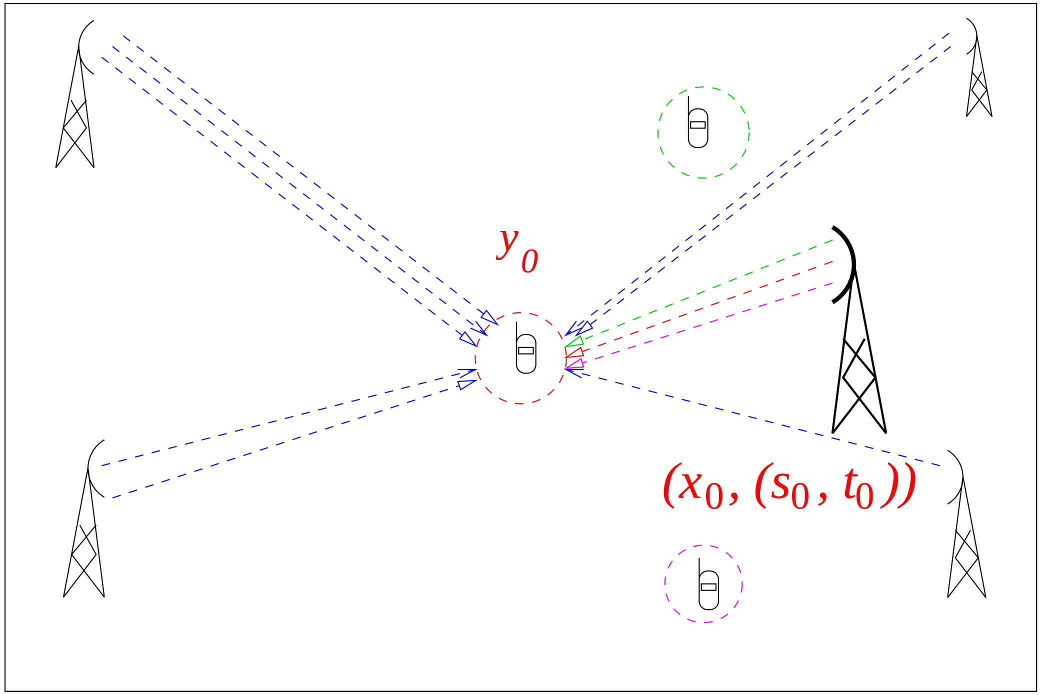
$$C_0(\Phi, W) = \{y : \frac{L(S_0, \theta_0, y - X_0)}{W + \kappa I_\phi(y)} \geq T_0\}$$

$L(S_0, \theta_0, y - X_0)$ takes into account the distance to the antenna and the orientation of the antenna (θ_0); I_ϕ defined similarly.

- Point dependent fading: for all y and all i, there is a random variable $Z_i(y)$ s.t. the power received at location y from the i th source is $S_i Z_i(y) l(y - X_i)$ in place of $S_i l(y - X_i)$.

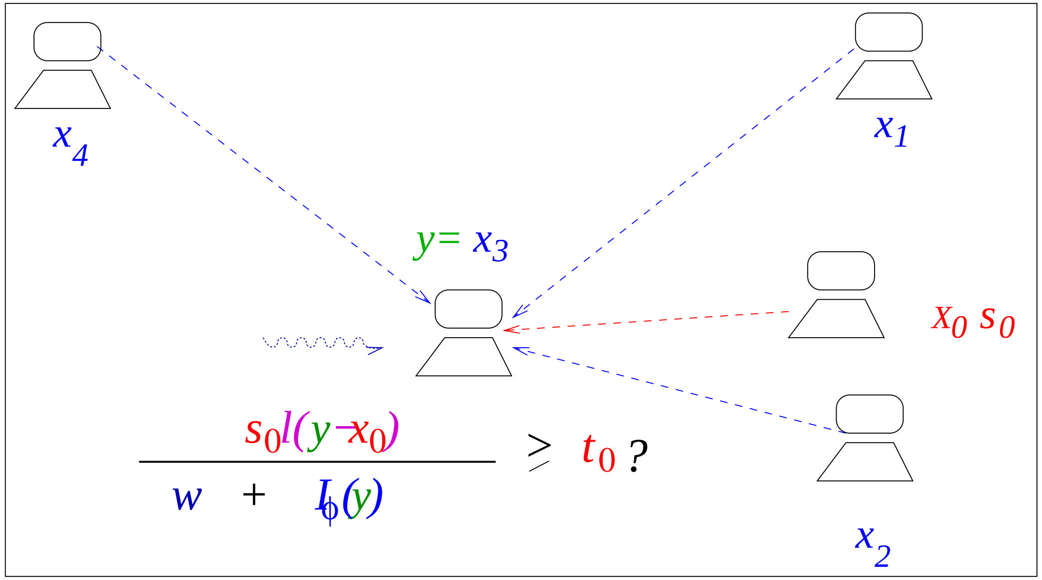
- Power control: to come in later lectures.

VARIANTS OF BASIC CDMA MODEL (*continued*)



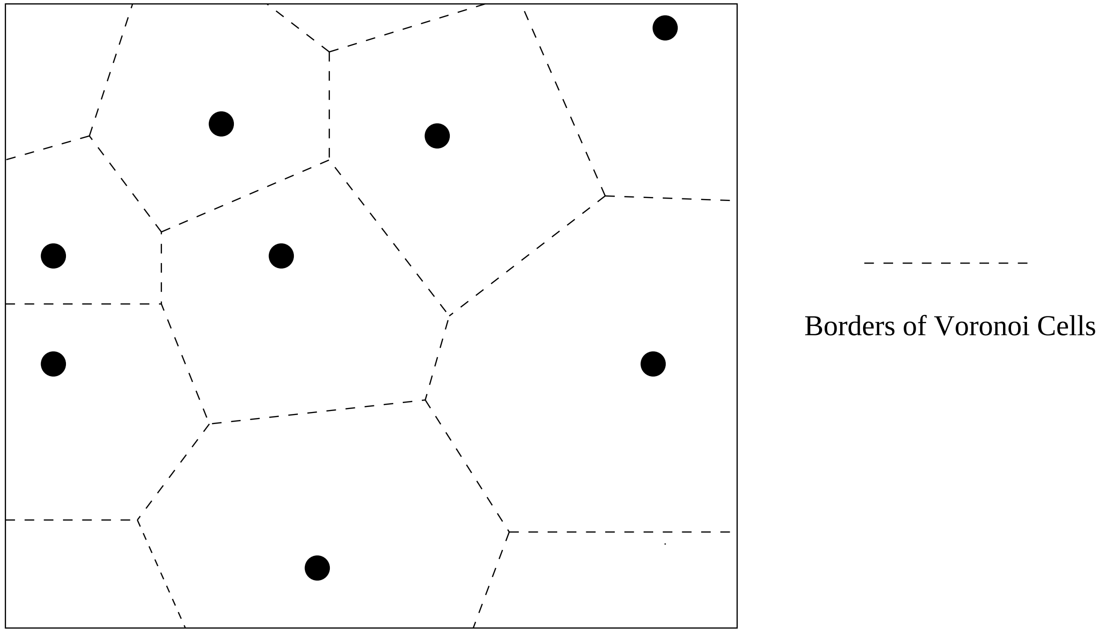
- Cluster point processes: possibly more than 1 points at X_i

MOTIVATION EXAMPLE 2: COMMUNICATION IN A MOBILE AD HOC NETWORK



VORONOI TESSELLATIONS

For X_0 a point of some spatial point process $\Pi = \{X_i\}$,
$y \in V_{X_0}$ if $||y - X_0|| \leq ||y - X_i||, \quad \forall i \neq 0$



- Each point y a.s. belongs to a single Voronoi cell;
- Spatial analogue of the intervals of renewal theory

BOOLEAN MODEL – GERM GRAIN MODEL

A Boolean model Ξ_{BM} is the union

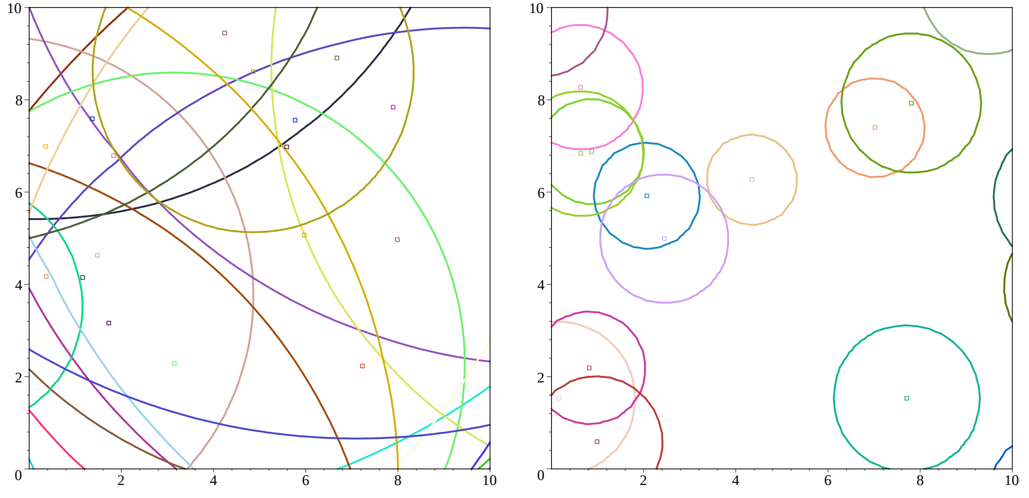
$$\Xi_{\mathrm{BM}} = \bigcup_i G_i \oplus X_i$$

- $\Phi = \{(X_i, G_i)\}$ is an independently marked Poisson p.p. with intensity measure ν on $\mathbb{R}^d$;
- $\{X_i\}$ points of the Poisson p.p. on $\mathbb{R}^d$;
- $\{G_i\}$ sequence of i.i.d random compact sets; $G \oplus x = \{y + x : y \in G\}$.

BOOLEAN MODEL – GERM GRAIN MODEL (*continued*)

Example

- $\{S_i\}$ sequence of integrable i.i.d. non-negative random variables, independent of the PPP;
- $b(u, x)$: closed ball centered in u and with radius $|x|$: $G_i = b(0, S_i)$.



RESULTS ON THE BOOLEAN MODEL

- The expected number of grains (sets $G_i \oplus X_i$) hitting a given bounded set B is

$$\mathbb{E}[\#\{(X_i, G_i) \in \Phi : B \cap G_i \oplus X_i \neq \emptyset\}] = \mathbb{E}[\nu(B \oplus \check{G})]\,,$$

  G is a generic (typical) random variable of the sequence $\{G_i\}$
  $B \oplus \check{G} = \{x + y : x \in B, -y \in G\}$.

- In the stationary case, proof based on Campbell's Theorem.

- Spatial analogue of the $M/GI/\infty$ queue.

- Example: homogeneous case with balls: Ξ is a random closed set if the radius of balls have moments of order d (2 in the plane);

RESULTS ON THE HOMOGENEOUS BOOLEAN MODEL

- The point process of the germs whose grain intersect compact K is Poisson of intensity measure of density

$$\lambda p(x) = \lambda \mathbb{P}(x + G \cap K \neq \emptyset).$$

- The distribution of the number of grains intersecting compact K is Poisson of parameter $\lambda \mathbb{E}|K \cap \check{G}|$.
- The number of grains covering location x is Poisson of parameter. $\lambda \mathbb{E}|G|$.
- The capacity functional of the coverage process is

$$T_{\Xi}(K) = \mathbb{P}(\Xi \cap K \neq \emptyset) = 1 - e^{-\lambda \mathbb{E}|K \cap \check{G}|}.$$

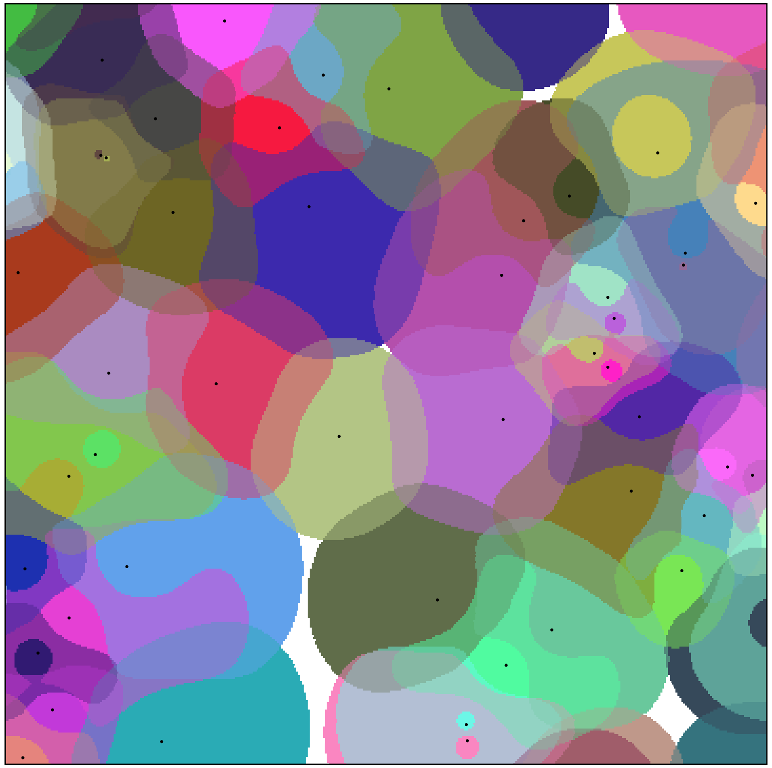
MATHEMATICAL QUESTIONS ON SINR COVERAGE

- Default option: non homogeneous Poisson point process with intensity measure ν, i.i.d. marks, $\kappa = 1$:

1. Is the model well-defined? Under some moment conditions for W and S, each particular cell $C_i = C_i(\Phi, , W)$ as well as the union Ξ are random closed sets.
2. What can we calculate?
   Probability for a typical cell to cover one or more locations (cell volume etc.);
   Distribution of the number of cells covering a given location (hand-off degree).
3. Relation to known models For some limit values of parameters, Ξ converges to a Boolean model or the Poisson-Voronoi tessellation + Expansions.

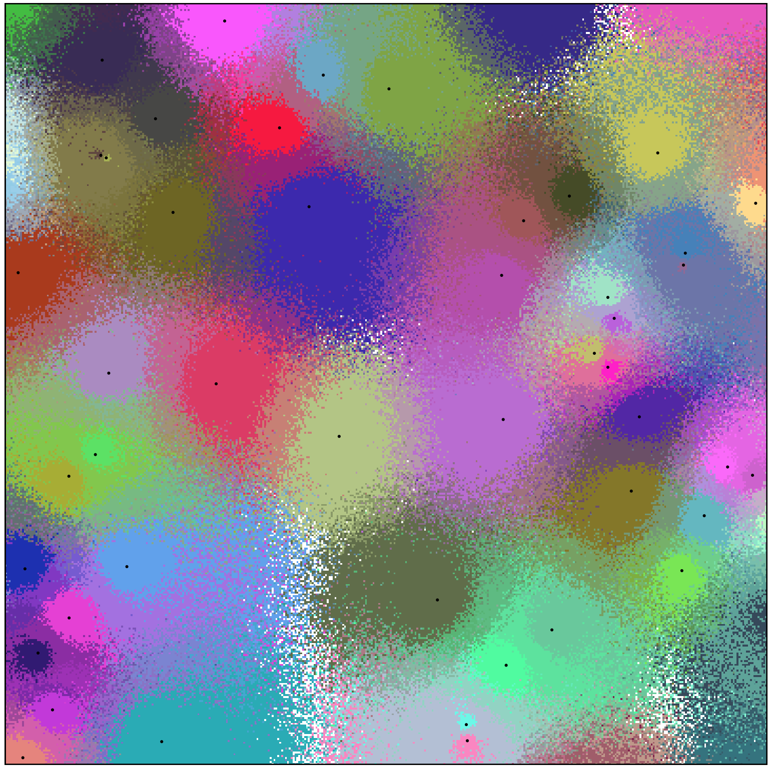
SIMULATION OF SINR-COVERAGE

cells without point dependent fading



[a]

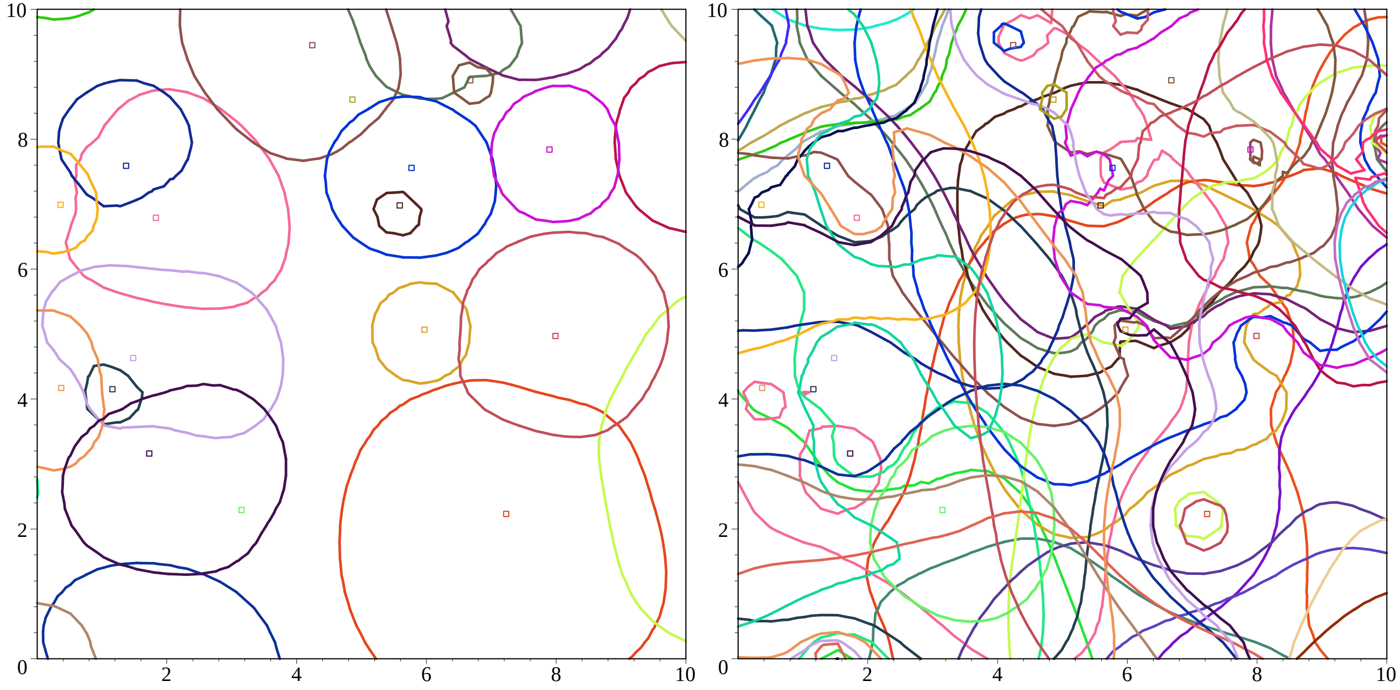
cells with point dependent fading



[b]

$\Phi = \{X_i, (S_i, T_i))\}$ — marked Poisson point process on $I\!R^2$.

SIMULATION OF SINR COVERAGE



RANDOM CLOSED SET CONDITIONS FOR A SINGLE CELL

A necessary and sufficient condition for the Shot Noise Process $I_\Phi(y)$ to have finite expectation is

$$\mathbb{E}[I_\Phi(y)] = \mathbb{E}[S] \int_{\mathbb{R}^d} l(y-x)\,\nu(dx) < \infty.$$

THEOREM

Suppose $l(\cdot)$ is continuous and for each $y \in \mathbb{R}^d$ $\exists$ a ball $b(y,\delta_y)$ s.t.

$$\int_{\mathbb{R}^d} \sup_{z \in b(y,\delta_y)} l(z-x)\,\nu(dx) < \infty\,.$$

If $\mathbb{E}[S] < \infty$, then with probability one the function

$$I_\Phi(y) = \sum_i S_i l(y - X_i)$$

is continuous with respect to y.

RANDOM CLOSED SET CONDITIONS FOR A SINGLE CELL *(continued)*

From Campbell's Theorem + Lebesgue dominated convergence theorem (extends to stationary point processes).

COROLLARY

Each particular cell

$$C_i = \{y : \frac{S_i l(y - X_i)}{W + I_\Phi(y)} \geq T_i\}$$

is a random closed set.

RANDOM CLOSED SET CONDITIONS FOR Ξ

Assumptions: homogeneous PPP; conditions of the previous theorem

THEOREM For all bounded sets B, $\mathbb{E}[\#\{C_i : B \cap C_i \neq \emptyset\}] < \infty$ and Ξ is a random closed set if one of the following conditions is satisfied:

(a) finite range of attenuation function: $l(z) = 0 \quad \text{for } |z| > R^*$.

(b) presence of noise W:

$$l(z) \leq a(1 + |z|)^{-\beta} \quad \text{for some } a, \beta > 0$$

and

$$\mathbb{E}[(S/T)^{d/\beta}] < \infty, \quad \mathbb{E}[W^{-d/\beta}] < \infty$$

(c) no noise W: $l(z)$ as above, $\mathbb{E}[S^{-d/\beta}] < \infty$ and for ach $R > 0$,

$$\int_{\mathbb{R}^d} e^{-\lambda b_d |x|^d} (\underline{l}(|x| + R))^{-d/\beta} \, dx < \infty \,, \quad \text{where} \quad \underline{l}(r) = \inf_{|z| \leq r} l(z).$$

RANDOM CLOSED SET CONDITIONS FOR Ξ (*continued*)

- In particular under any of the above assumptions, K_y, the number of cells covering point y, has finite expectation.

IDEA OF PROOF

$$C_0 = \{y : \frac{s_0 l(y-x_0)}{w + I_\phi(y)} \geq t_0\}$$

$$\subset \{y : l(y-x_0) \geq \frac{w}{s_0/t_0}\}$$

$$\subset \{y : |y-x_0| \leq (\frac{a s_0/t_0}{w})^{1/\beta}\} = b(x_0, (\frac{a s_0/t_0}{w})^{1/\beta}) \, .$$

$$C_i \subset b(X_i, \rho_i) \, , \quad \text{with} \quad \rho_i = (\frac{a S_i/T_i}{W})^{1/\beta} \, .$$

PROBABILITY FOR A TYPICAL CELL TO COVER A POINT

- Φ marked Poisson point process representing antennas in $\mathbb{R}^d$,
- $(x, (S, T))$ additional antenna located at fixed location x with random mark (S, T) distributed as any mark of Φ, independent of it,
- y location (of a customer) in $\mathbb{R}^d$.

Probability of covering y by the cell attached to the additional antenna:

$$\begin{aligned} p_x(y) &= \mathbb{P}(\, y \in C(x, S, T; \Phi, W)\,) \\ &= \mathbb{P}(\, \frac{S}{T} l(y - x) - W - I_\Phi(y) \geq 0\,)\,. \end{aligned}$$

M/GI Case

- $p_x(y)$ can be obtained via a singular contour integral from the Laplace transforms of the RV's (S,T), W and $I_\Phi(y)$ which are independent (Slyvniak's th.);

THEOREM

- the Laplace transform of $I_\Phi(y)$ is given by

$$\Psi_{I_\Phi(y)}(\xi) = I\!E[\exp(-\xi I_\Phi(y))] = \exp[-\int_{I\!R^d}(1-\Psi_S(\xi l(y-z)))\,\nu(dz)]\,,$$

where $\Psi_S(\xi) = I\!E[e^{-\xi S}]$ is the Laplace transform of S

- Integral representation: if the real valued random variable Y has a density and Fourier transform $\psi(\xi) = I\!E[\exp(-i\xi Y)]$, $\xi \in I\!R$, then

$$I\!P(Y \geq 0) = \frac{1}{2} - \frac{1}{2i\pi}\int_{I\!R}\frac{\psi(\xi)}{\xi}\,d\xi,$$

M/M Case

- Exponential powers with parameter μ

$$
\begin{aligned}
p_x(y) &= I\!P(\, S \ge \frac{T}{l(y-x)}(W + I_\Phi(y))\,) \\
&= \int_t \int_u e^{-\mu \frac{ut}{l(y-x)}} I\!P(\, W + I_\Phi(y) = du\,) I\!P(\, T = dt\,) \\
&= \int_t \Psi_W\left(\frac{\mu t}{l(y-x)}\right) \Psi_{I_\Phi(y)}\left(\frac{\mu t}{l(y-x)}\right) I\!P(\, T = dt\,)
\end{aligned}
$$

- For constant threshold T and exponential power:

$$
p_x(y) = \Psi_W\left(\frac{\mu T}{l(y-x)}\right) \Psi_{I_\Phi(y)}\left(\frac{\mu T}{l(y-x)}\right).
$$

- For constant threshold T, exponential power homogeneous PPP:

$$
p_0(y) = \Psi_W\left(\frac{\mu T}{l(y)}\right) \exp\left\{-2\pi\lambda \int_0^\infty \frac{u}{1 + l(y)/(Tl(u))} du\right\}.
$$

SCENARIO 1

- $d = 2$ and $\nu(x) \equiv 1$;
- $l(z) = (\max(|z|, R)^{-4}$
- $S \in I\!R^+$ exponential with mean $m = 1/\mu$;

$$\psi_{I_\Phi}(\xi i) = I\!E[e^{-i\xi I_\Phi}] = \exp\left[\pi\sqrt{\frac{i\xi}{m}}\arctan(R^2\sqrt{\frac{m}{i\xi}}) - \frac{1}{2}\pi^2\sqrt{\frac{i\xi}{m}}\right],$$

SCENARIO 2

– $d = 2$ and $\nu(x) \equiv \lambda dx$;

– $l(z) = |z|^{-\beta}$, $W \equiv 0$;

– T deterministic and S exponential with mean $m = 1/\mu$;

If $\beta > 2$

$$\int_0^\infty \frac{u}{1 + l(y)/(Tl(u))}\, du \;=\; \frac{|y|^2 T^{2/\beta}}{\beta}\Gamma(2/\beta)\Gamma(1-2/\beta)\,,$$

Hence

$$p_0(y) = e^{-\lambda |y|^2 T^{2/\beta} K}\,,$$

with $K = K(\beta) = (2\pi\Gamma(2/\beta)\Gamma(1-2/\beta))/\beta$.

MEAN CELL SIZE

- $\mathcal{A}(x) = \mathcal{A}(C(x, S, T; \Phi, W))$: volume of the cell added at point x.
- Mean volume in the M/GI case:

$$E[\mathcal{A}(x)] = E \int_{\mathbb{R}^d} 1_{y \in C(x)} dy = \int_{\mathbb{R}^d} p_x(y) dy.$$

- Mean volume in the M/M case of Scenario 2:

$$E[\mathcal{A}(0)] = 2\pi \int_0^\infty e^{-\lambda r^2 T^{2/\beta} K} r dr = \frac{1}{\lambda T^{2/\beta}} \frac{\beta}{2\Gamma(2/\beta)\Gamma(1-2/\beta)}.$$

PROBABILITY FOR CELLS TO COVER POINTS

- (y_1, y_2) locations to be respectively covered by the cells $C(x_1, S_1, T_1; \Phi + \delta_{x_2,S_2,T_2}, W)$ and $C(x_2, S_2, T_2; \Phi + \delta_{x_1,S_1,T_1}, W)$
- Similar analysis from the joint Laplace transform of $(I_\Phi(y_1), I_\Phi(y_2))$, which is given by

$$\mathbb{E}[\exp(-\xi_1 I_\Phi(y_1) - \xi_2 I_\Phi(y_2))] =$$
$$\exp[-\int_{\mathbb{R}^d}(1 - \Psi_S(\xi_1 l(y_1 - z) + \xi_2 l(y_2 - z))\nu(dz)] \, .$$

OVERLAPPING CELLS

- Given: n cells $C(x_i, s_i, t_i; \phi, w)$, $i = 1, \ldots, n$

THEOREM The inequality

$$\sum_{i=1}^{n} t_i < 1$$

is a necessary condition for $\cap_{i=1}^{n} C(x_i, s_i, t_i; \phi, w) \neq \emptyset$.

PROOF The set of inequalities

$$\frac{s_i l(y - x_i)}{w + \Sigma_{i=1}^{n} s_i l(y - x_i)} \geq t_i \quad (i = 1, \ldots, n)$$

implies

$$1 - \frac{w}{w + \Sigma_{i=1}^{n} s_i l(y - x_i)} \geq \sum_{i=1}^{n} t_i$$

and for $w > 0$ the LHS is strictly less than 1.

OVERLAPPING CELLS (*continued*)

COROLLARY If the distribution of T is such that

$$T \geq \tau \qquad \text{a.s, for some } \tau > 0,$$

then the number K_y of cells of Ξ covering any location y is a.s. bounded

$$K_y < 1/\tau.$$

- Spatial analogue of s server queue: no location can be covered by $s = 1/\tau$ or more cells, no matter how close they are and how strong their signals are.

MOMENT EXPANSION OF THE NUMBER OF CELLS K_y COVERING y

THEOREM

The factorial moment of K_y : $\mathbb{E}[K_y^{(n)}] = \mathbb{E}[K_y(K_y-1)\dots(K_y-n+1)_+]$ is given by:

$$\int_{(\mathbb{R}^d)^n} \mathbb{P}(y \in \bigcap_{k=1}^{n} C(x_k, S_k, T_k; \Phi + \sum_{\substack{i=1 \\ i\neq k}}^{n} \varepsilon_{(x_i,(S_i,T_i))}, W))\nu(dx_1)\dots\nu(dx_n)\ .$$

- Computational aspect: $F_{I_\Phi(x)}(\cdot)$: distribution function of the shot-noise process at x:

$$\mathbb{P}(y \in \bigcap_k C(\dots)) = \mathbb{E}[F_{I_\Phi(y)}(\min_{1\le k\le n} S_k/T_k\, l(y-x_k) - \sum_{k=1}^{n} S_k l(y-x_k) - W)].$$

PROOF

$$K_x^{(n)} = \mathbb{E} \int_{(\mathbb{R}^d\times\mathbb{R}^+\times(0,1))^n} \prod_{k=1}^{n} \mathbf{1}\left(x \in C(x_k, s_k, t_k; \Phi - \varepsilon_{x_k,(s_k,t_k)}), W\right)$$
$$\times\Phi^{(n)}\left(d\left((x_1,\ldots,x_n),((s_1,\ldots,s_n),(t_1,\ldots,t_n)\right)\right)\right)$$

with $\Phi^{(n)}$ the n-th factorial power of Φ.

- Apply the refined Campbell theorem to the expectation of this integral and the fact that the reduced Palm distribution of the Poisson p.p. is equal to the original distribution.

HOMOGENEOUS CASE

- For a homogeneous Poisson point process with intensity λ

$$\mathbb{E}[K_0] = \lambda \mathbb{E}[\mathcal{A}(C(0, S, T; \Phi, W))]\,,$$

where $\mathcal{A}(C(\ldots))$ is the surface of the typical cell: Analogue of Little's law.

- The volume fraction p (fraction of the space covered by Ξ) is given by

$$p = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \mathbb{E}[(K_0)^{(k)}]$$

whenever $T \geq \tau$ a.s. (then the summation is over $0 \leq k < 1/\tau$).

QUALITATIVE RESULTS: CONVERGENCE TO A BOOLEAN MODEL

- Ξ^κ the coverage process generated by the Poisson point process

$$\Phi = \{(X_i, (S_i, T_i))\}$$

  in the presence of the orthogonality factor κ.

- Basic observation the sets

$$C_i^\kappa = \{y : S_i\, l(y - X_i) \geq WT_i + I_\Phi(y)\, T_i\kappa\}$$

  are increasing when $\kappa \longrightarrow 0$.

- Equivalent formulation with $\kappa \equiv 1$: $W \longrightarrow \infty$ and $T \longrightarrow 0$ in such a way that $WT = cst$.

QUALITATIVE RESULTS: CONVERGENCE TO A BOOLEAN MODEL (*continued*)

THEOREM

$$\Xi^{\kappa} \quad \xrightarrow{\kappa \to 0} \quad \tilde{\Xi} \quad \text{a.s}$$

where

$$\tilde{\Xi} = \tilde{\Xi}(\Phi; W) = \bigcup_{(X_i,(S_i,T_i))\in\Phi} \tilde{C}(X_i, S_i, T_i; W)$$

is a conditional Boolean model with cells

$$\tilde{C}(X_i, S_i, T_i; w) = \{y \in I\!R^d : S_i\, l(y - X_i) \geq wT_i\}$$

independent given $W = w$.

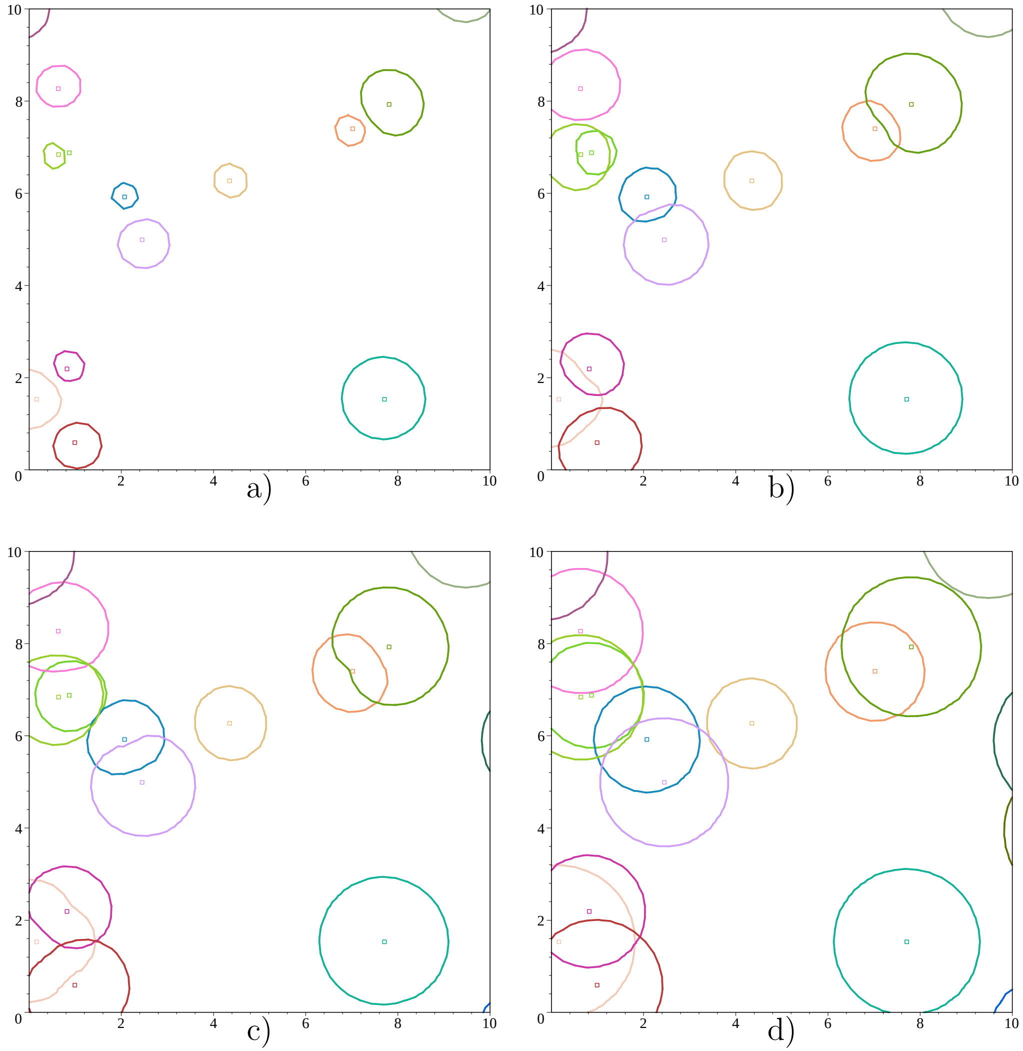
- Under regularity conditions on $l(\cdot)$, we also have

– convergence on the space of closed sets,

– convergence of the capacity functionals (of the typical cell and the union).

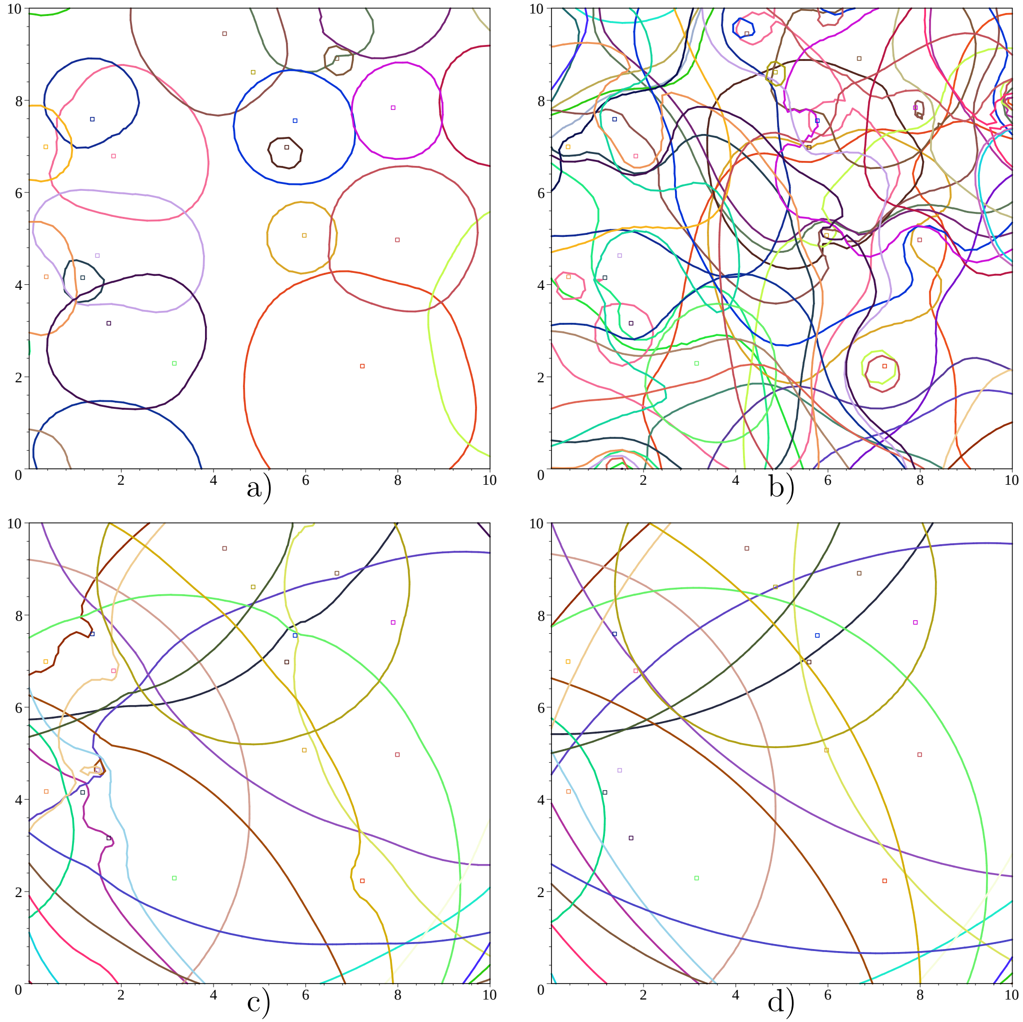
SCENARIO 3

- Simulation of the coverage process $\Xi(\Phi; W)$ "on its way" to a Boolean model
- There are 60 points on the square $[-5, 15]^2$ (making $\lambda = 0.15$) with S uniformly distributed on $[0, 2]$
- The observation window is $[0, 10]^2$,
- $l(y) = (1 + |y|)^{-3}$,
- a) $T = 0.4$, $W = 0.25$, b) $T = 0.2$, $W = 0.5$, c) $T = 0.1$, $W = 1$; d) $T = 0.0001$, $W = 1000$
- In the limiting case, each cell is a disk with independent radius distributed as $(S/(WT))^{1/3} - 1 = (10S)^{1/3} - 1$ with the mean $\approx$ 1.035.


a)
b)
c)
d)
0
2
4
6
8
10

SCENARIO 4

- a) $T = 0.2$, $W = (0.1)^3$, b) $T = 0.2 \cdot 10^{-2}$, $W = (0.1)^2$, c) $T = 0.4 \cdot 10^{-4}$, $W = 5$; d) $T = 0.2 \cdot 10^{-5}$, $W = 100$
- In the limiting case, each cell is a disk with independent radius distributed as $(S/(WT))^{1/3} - 1 = (5000S)^{1/3} - 1$ with mean $\approx$ 16.15.


a)
b)
c)
d)

COROLLARY

$$p_x^{(\kappa)}(y) = I\!P(S_0 l(y-x) \geq WT_0 + \kappa T_0 I_\Phi(y))\,.$$

- Continuity result under technical conditions,

$$p_x^{(\kappa)}(y) = I\!P(S_0 l(y-x) \geq WT_0) + o(1), \quad \kappa \to 0.$$

PERTURBATION FORMULAS

- Perturbation formula (assuming $T_i > 0$ a.s.) Let

$$F_*(u) = \mathbb{P}(\frac{S_0}{T_0} l(y-x) - W < u)\,.$$

If F_* admits the following expansion at 0:

$$F_*(u) = F_*(0) + \sum_{k=1}^{h} \frac{F_*^{(k)}(0)}{k!} u^k + \mathcal{R}(u) \quad \text{and} \quad \mathcal{R}(u) = o(u^h) \quad u \searrow 0.$$

Then

$$p_x^{(\kappa)}(y) = \mathbb{P}(S_0 l(y-x) \geq W T_0) - \sum_{k=1}^{h} \kappa^k \frac{F_*^{(k)}(0)}{k!} \mathbb{E}[(I_\Phi(y))^k] + o(\kappa^h)\,,$$

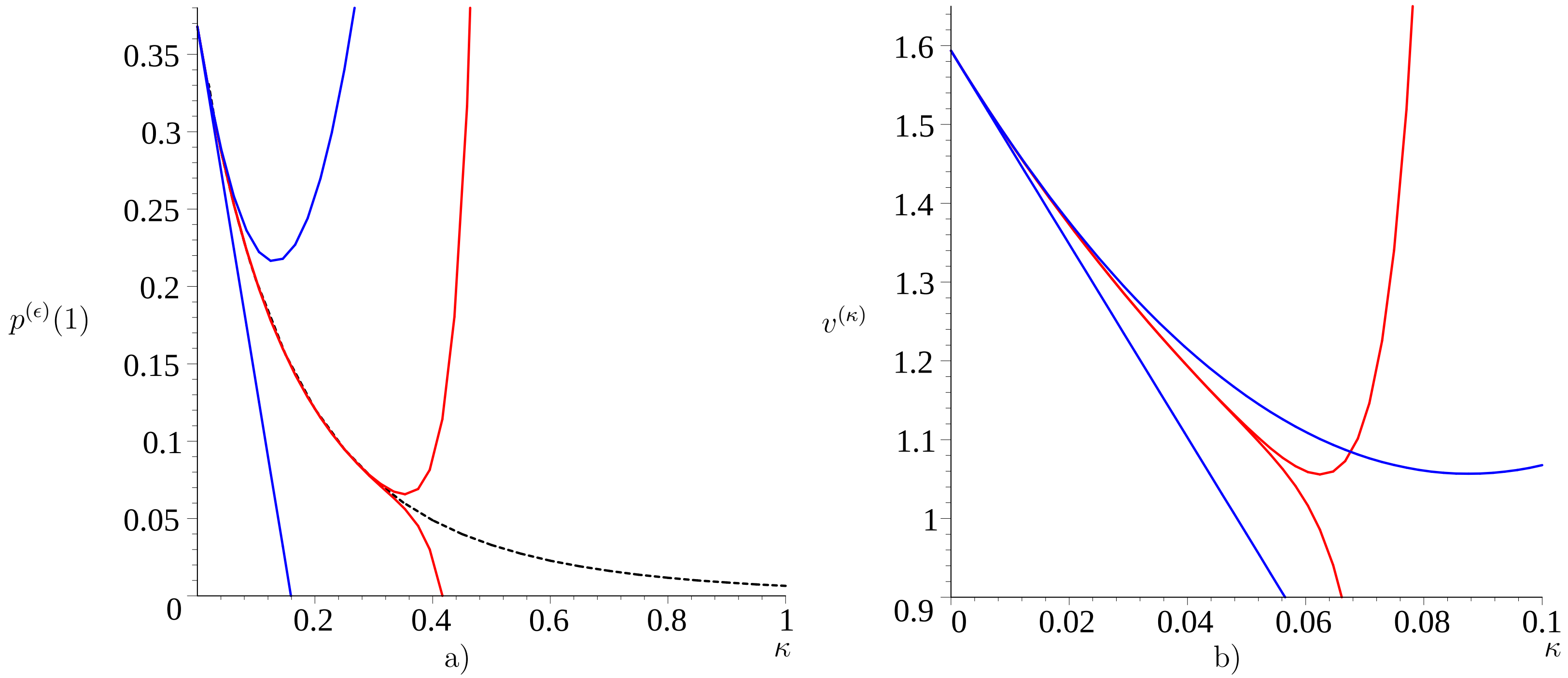
provided $\mathbb{E}[(I_\Phi(y))^h] < \infty$.

PERTURBATION FORMULAS (*continued*)

- Idea of proof (order 1):

$$
\begin{aligned}
& \mathbb{P}(S_0 l(y-x) \geq W T_0 + \kappa T_0 I_\Phi(y)) \\
= \; & \mathbb{P}(S_0 l(y-x) \geq W T_0) - \mathbb{P}(0 \leq \frac{S_0}{T_0} l(y-x) - W < \kappa I_\Phi(y)) \\
= \; & \mathbb{P}(S_0 l(y-x) \geq W T_0) - \mathbb{E}(F_*(\kappa I_\Phi(y)) - F_*(0)) \\
= \; & \mathbb{P}(S_0 l(y-x) \geq W T_0) - \kappa \mathbb{E}[(I_\Phi(y))] F_*^{(1)}(0) + o(\kappa).
\end{aligned}
$$

SCENARIO 1 (CONTINUED)



a) Exact values of $p_x(y)$ (dashed line, obtained from the singular integral representation) and the first, second, 14-th and 15-th order approximation of $p_x^{(\kappa)}(y)$. b) Similar approximation for the mean area of the typical cell.

CONVERGENCE TO THE POISSON VORONOI TESSELLATION

- $\Xi^n = \cup_i C_i^n$ coverage process generated by the Poisson point process with attenuation function

$$l_n(z) = (1 + |z|)^{-n}$$

and with $W = 0$ a.s.

THEOREM

$$C_i^n \quad \stackrel{n\to\infty}{\longrightarrow} \quad V_i \quad \text{a.s}$$

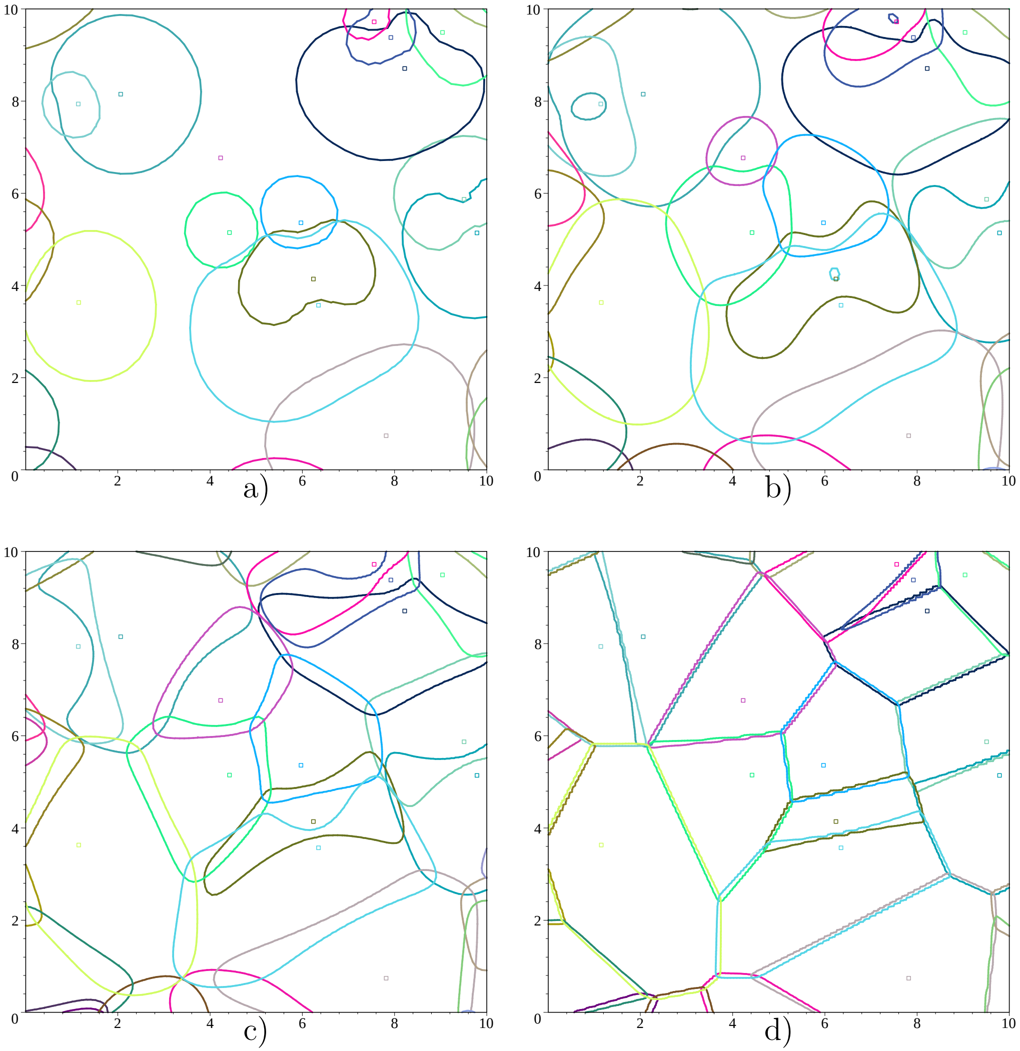
where

$$V_i = \{y \in \mathbb{R}^d : |y - X_i| \le \inf_{X_k \in \Phi - \varepsilon_{X_i}} |y - X_k|\}$$

is the Voronoi cell generated by the point X_i of Φ.
Convergence holds on the space of closed sets.
We also have convergence of the volume of the typical cell.

SCENARIO 5

- Simulation of the coverage process Ξ tending to Voronoi tessellation of the plane
- Same window as above
- $W = 0$ and $T = 0.2$ (allowing for intersections).
- $l(y) = (1 + |y|)^{-\beta}$ with: a) $\beta = 3$, b) $\beta = 5$, c) $\beta = 12$, d) $\beta = 100$.
- The effect of overlapping is still visible.


a)
b)
c)
d)

RELATION TO THE JOHNSON-MEHL MODEL

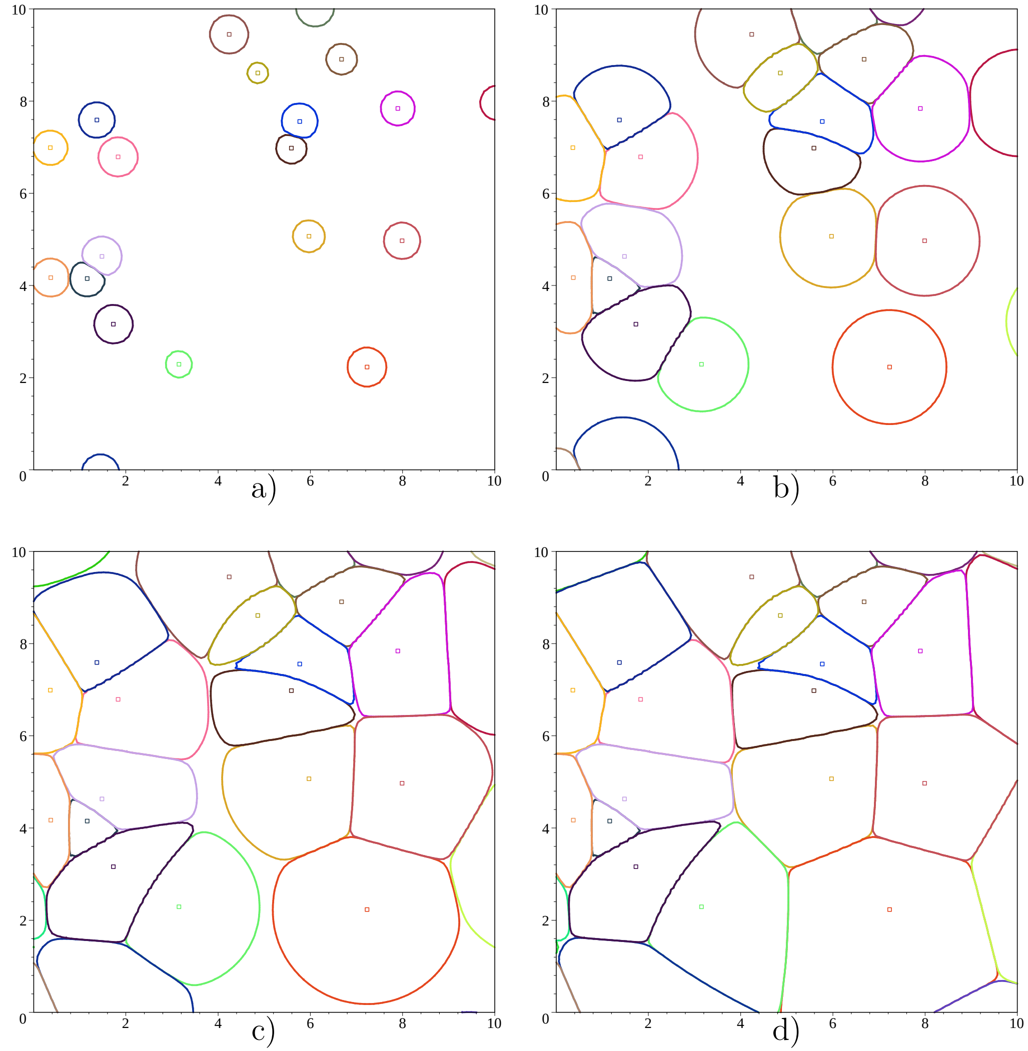
COROLLARY If $W = (R+1)^{-n}$ then

$$C_i^n \xrightarrow{n\to\infty} V_i \cap b(X_i, R)$$

where $b(x,r)$ is the ball centered at x with radius r.

SCENARIO 6

- Simulation of the coverage process Ξ growing to the tessellation of the plane as in the Johnson-Mehl model;
- Window as above;
- We take $T = 0.5$, thus inhibiting any intersections;
- $l(y) = (1 + |y|)^{-30}$, strong enough to give the tessellation covering almost the whole plane when there is no external noise W;
- $W = (1 + R)^{-30}$
- a) $R = 0.4$, b) $R = 1.2$, c) $R = 2$, d) $R = \infty$ ($W = 0$).
- All cells start growing at the same time.


a)
b)
c)
d)
0
2
4
6
8
10

CONCLUSIONS – SG

- New basic parametric SG model
- Contains well known models of SG as particular cases
- Corresponds to a sharing of space which is a spatial analogue of multiserver queues
- Questions to be addressed later:

1. infinite components,
2. local interactions,
3. geometry of zones defined by their degree of coverage,

FURTHER STEPS IN RELATION WITH WIRELESS NETWORKS

– Power control: S_i may be seen as random at any given time, but in fact results from joint adaptive schemes; it would make more sense to have dependencies between the marks of cells;

– Analysis of traffic: Coverage varies with time: dynamical aspects

– Routing protocols, MAC protocols etc.